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TEXT, CONTEXT, SUBTEXT

BY

MICHAEL N. FRIED
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PRINTED IN THE NETHERLANDS

To Michael's parents, Dan and Thelma

&

To the memory of Sabetai's, Zeida and Ghitel

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PREFACE AND ACKNOWLEDGEMENTS

This is a book about conic sections in Antiquity and, specifically, about Apollonius's *Conica.* It presents a rival interpretation to Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*, the premier treatment of the topic since its appearance in 1886. While Zeuthen's title is straightforward (which did not prevent him from using generously distorting textual glosses), ours may need a brief clarification. In writing this book, Apollonius's text has been the mainstay of all our readings and commentaries. It is the text as transmitted that guided our exposition and elucidation, the text itself, and not its various algebraic transmutations. In dealing with the text, we strove to use for its understanding only those mathematical texts forming its immediate Greek mathematical context. We see in this the proper historical procedure in rendering the history of mathematics for the contemporary reader.

We do not believe there is any hidden agenda to the text of the *Conica.* The text is all we have (that's all Zeuthen, or anybody else, ever had), there is no accompanying hidden, algebraic subtext presenting the historian with *the* key for the proper understanding of the text. What lies behind the text are the Greek mathematical texts of Apollonius's predecessors and nothing else. If these can be considered forming a subtext, then it is the only subtext we are willing to accept. This, then, explains the subtitle of our book.[1]

There are a number of scholarly institutions where one of us spent sabbaticals of various lengths and where he pondered and worked on Apollonius over the years. They all have earned our gratitude. In chronological order they are: Herzog August Bibliothek, Wolfenbüttel, its late Direktor and Leiterin der Forschungsabteilung, Paul Raabe and Sabine Solf respectively, the Wissenschaftskolleg zu Berlin, its Rektor and Sekretär, Wolf Lepenies and Joachim Nettelbeck, the Max Planck Institut für Wissenscaftsgeschichte, Berlin and its Direktor, Jürgen Renn, the Collegium Budapest and its two consecutive Directors,

[1] See Sabetai Unguru, "Le *Coniche* di Apollonio di Perga Testo, Contesto, Sottotesto," *Conferenze e Seminari 1999–2000*, Mathesis (Torino, 2000), pp. 251–262.

Lajos Vékás and Gábor Klaniczay, The European Humanities Research Centre, Oxford and its two Directors, Helen Watanabe-O'Kelly and Ian MacLean, and the Department of Mathematics of the University of Torino, its Chairman, Franco Pastrone and Professor Livia Giacardi.

The Cohn Institute for the History and Philosophy of Science and Ideas at Tel-Aviv University, where one of us earned his Ph. D. writing on the *Conica*, its present Director, Rivka Feldhay, as well as her predecessors, also merit our gratitude.

This brings us to individuals. First, we would like to thank one another. The thanks are well deserved. But there are also others who deserve our thanks: John Neu, for supplying us with a copy of Halley's reconstruction of the lost,eighth, book of the *Conica*; Jan P. Hogendijk, for a partial copy of his edition of Ibn al Haytham's *Completion of the Conics*; ChristianMarinusTaisbak, for a copy of Olaf Schmidt's article on Archimedes and for his congeniality and permanent readiness to help; David C. Lindberg, the mentor of one of us, for his continuous interest, support, and friendship; Arnold Thackray, who, as Editor of *Isis*, went out on a limb to stop the muzzling of one of us by prominent proponents of "Geometrical Algebra"; Bill Donahue, for his helpful comments on the translation of Book IV of the *Conica*, and Dana Densmore and Bill Donahue for agreeing to the publication of a full-fledged scholarly edition of that translation in our present book; Miriam Amit at the Center for Science and Technology Education at Ben-Gurion University of the Negev for lightening the load of one of us; Orna Harari-Eshel, for supplying us with a set of Greek fonts; finally, חביב, אחרון ,אחרון, Yifat and Yochi, for their unswerving support and, most importantly, for just being there.

Michael N. Fried, Sabetai Unguru

INTRODUCTION

Since H. G. Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*, written in 1886, there has appeared no other major interpretation of Apollonius's *Conica*. Even Heath's book-size introduction to his *Apollonius of Perga: Treatise on Conic Sections* is really only an adaptation of Zeuthen's work. Indeed, almost every modern work on Apollonius's *Conica*, short or long, either has been a mere summary of Zeuthen's views or an account deeply influenced by them. This, of course, is not to say that in the hundred odd years since the publication of *Die Lehre* there have been no important contributions to Apollonius scholarship. No one can deny the importance, for example, of Toomer's translation of the later books of the *Conica* from the Arabic. Yet, here too, one strongly feels Zeuthen's presence, not only in Toomer's "Mathematical Summary" and in his commentary, but also in his free use of modern algebraic notation in the translation itself. The purpose of this book, then, is to present a new interpretation of the *Conica*, an alternative to Zeuthen's reading of the text.

But is a wholly new interpretation truly warranted? It is for two reasons. The first has to do with the significance of Apollonius and the *Conica* in the history of mathematics, and the second, with the character and foundation of Zeuthen's interpretation. Indeed, if the *Conica* were only an obscure work of an obscure author, a new interpretation of it, even if that interpretation were to set aright old *mis*-interpretations, might justly be seen as a mere exercise in scholarly esoterica. But Apollonius of Perga is no unknown and insignificant figure: together with Euclid and Archimedes, he towers over Hellenistic mathematics, and of his works the *Conica* is the most famous, as well as one of the most important in the entire corpus of Greek mathematics. For this reason, Eutocius relates that, according to Geminus, it was because of "the remarkable nature of the theorems in conics proved by him" that Apollonius was known as "The Great Geometer." The understanding of the *Conica*, therefore, must have immense weight in the *general* understanding of Greek mathematics, especially in the Hellenistic period, its golden age. Before we present the plan of the present book, then, let us survey the history of the *Conica*.

Apollonius's Conica *and its History*

About Apollonius the man, one can at best sketch a rough outline of his life. According to Eutocius, who is himself relying on another source, Heraclius,[1] Apollonius was born in Perga[2] in Pamphylia in the time of Ptolemy Euergetes.[3] Since Ptolemy Euergetes' reign was from 246–221 B.C.E., while Ptolemaeus Chennus refers to a famous astronomer, Apollonius (presumably, *our* Apollonius), in the time of Ptolemy Philopator (221–205 B.C.E.), it is reasonable to assume that Apollonius was born sometime around 240 B.C.E.[4] From the letters introducing the books of the *Conica*, we know that Apollonius resided at one time or another in Ephesus, Pergamum, and Alexandria, though, "we must confess that we cannot know where (if anywhere) his permanent domicile was."[5] Pappus claims that Apollonius actually *studied* in Alexandria under students of Euclid.[6] Writers have often repeated Pappus's claim as if it were a well established fact,

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[1] We are not completely sure who this Heraclius is, but Knorr claims that he is, in fact, the same as the biographer of Archimedes, Heraclides, whom Eutocius mentions in his commentary on Archimedes. Knorr argues that, "As we have seen, [Heraclides] composed a biographical account of Archimedes which included an apologia for Archimedes' circle measurement. Now, in his opening remarks in the commentary on Apollonius' *Conics*, Eutocius draws from an Archimedes biography a statement of the relative contributions of Archimedes and Apollonius to the foundation of the theory of conics. In our manuscripts the biographer's name appears as 'Heraclius' (Greek: *Herakleios*), but one immediately suspects a scribal slip for 'Heraclides' (*Herakleides*); for not only is it unlikely that two different individuals with such closely resembling names should both have composed biographies of Archimedes, but also that they share the polemical attitude with respect to the work of Apollonius" (*Ancient Tradition of Geometric Problems*, pp. 297–98).

[2] Presumably, this is the Greek city of Perge, which was located near what is today the city of Antalya on the southern coast of Turkey.

[3] Eutocius, *Commentaria in Conica*, Heiberg (*Apollonii Pergaei*), II,168.

[4] This is the date Toomer gives in his *Apollonius Conics Books V to VII* (p. xi), Heath gives 262 B.C.E. (*i.e.* twenty-five years after Archimedes) in his *History of Greek Mathematics* (II,126). We tend to accept the later date of 240 B.C.E for two reasons: 1) Eutocius (*In Conica*, p. 168) says that Apollonius was *born* (γέγονε) (Heath has "flourished") in Perge in the time of Ptolemy Eurgetes and 2) other arguments place the writing of the *Conica* sometime in the early part of the second century B.C.E. (see below).

[5] Toomer, *Apollonius Conics Books V to VII*, p. xii.

[6] Pappus, *Coll.*, VII (Hultsch, p. 678). One should note that Pappus makes the remark about Apollonius's studying under Euclid's successors in a rather vituperative passage in which he claims that Apollonius did not give Euclid enough credit in the solution of the three and four line locus.

though that is far from clear.[7] What we do know, by his own account in the prefatory letter to Book I of the *Conica*, is that Apollonius composed the initial version of the *Conica* while he was in Alexandria, and, from Apollonius's mentioning the geometer Philonides (in the prefatory letter to Book II) Toomer suggests that this was sometime after 200 B.C.E. Toomer argues as follows: "Philonides, as we learn from a fragmentary biography preserved on a papyrus and from two inscriptions, was an Epicurean mathematician and philosopher who was personally known to the Seleucid kings Antiochus IV Epiphanes (reigned 175–163 B.C.) and Demetrius I Soter (162–150 B.C.). Eudemus [to whom the letters introducing Books I and II of the *Conica* are addressed] was the first teacher of Philonides. Thus the introduction of the young Philonides to Eudemus [which Apollonius mentions in the preface to Book II] probably took place early in the second century B.C. The *Conics* were composed about the same time."[8]

Besides the *Conica*, what we know of Apollonius's other mathematical works can be summarized briefly as follows.[9] Pappus cites and discusses six other works by Apollonius:[10] *Cutting off of a Ratio* (Λόγου Ἀποτομή) in two books, *Cutting off of an Area* (Χωρίου Ἀποτομή) in two books, *Determinate Section* (Διωρωσμένη Τομή) in two books, *Tangencies* (Ἐπαφαί) in two books, *Neusis* (Νεύσεις) in two books, and *Plane Loci* (Τόποι Ἐπίπεδοι) also in two books. Of these, we have only *Cutting off of a Ratio*, and that, in an Arabic translation (Halley translated this work into Latin in 1704, six years before his translation of the *Conica*). Pappus also mentions in the second book of his

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[7] See Toomer, "Apollonius of Perga," p. 179, *Apollonius Conics Books V to VII*, p. xii and A. Jones, *Pappus of Alexandria: Book 7 of the Collection*, II, 402–403.

[8] Toomer, "Apollonius of Perga," p. 179.

[9] A more complete description of these works can be found in Heath, *History*, II, 175–196. One work listed by Heath that we have omitted is *On the Burning Mirror* (Περὶ τοῦ πυρίου). This work has indeed been attributed traditionally to Apollonius because of its mention in the Bobbio Mathematical Fragment (Heiberg, *Mathematici Graeci Minores*, p. 88). Based largely on his discovery of an Arabic translation of Diocles' work *On Burning Mirrors* (Περὶ πυρίων), Toomer has given reasons to ". . . consider it highly probable that the author [of the Bobbio fragment] had the work of Diocles in his hands, and misattributed it to Apollonius (there is no other evidence for a work of this title by Apollonius)" (Toomer, *Diocles On Burning Mirrors*, p. 20).

[10] *Coll.*, VII, Hultsch, p. 636.

Collection[11] a method developed by Apollonius in representing and multiplying large numbers. Pappus, however, does not give the name of this arithmetic work. In the work of the tenth century A.D. Arabic geometers, Ibrahim ibn Sinan and Al-Sījzi, Hogendijk[12] has found evidence that these six works mentioned by Pappus were all translated into Arabic, and, has discovered in Ibrahim ibn Sinan and Al-Suzi's writings fragments from the *Plane Loci* and *Neusis*. Hypsicles[13] mentions in his preface to the spurious Book XIV of Euclid's *Elements* another work by Apollonius, *On the Comparison of the Dodecahedron and the Icosahedron* (Συγκρίσεως τοῦ δωδεκαέδρου πρὸς τὸ εἰκοσάεδρον), in which, among other things, Apollonius is said to have proven that if the solids are inscribed in the same sphere then their surfaces have the same ratio as their volumes. A work by Apollonius apparently on the fundamental principles of mathematics,[14] the *General Treatise* (Καθόλου πραγματεία), is mentioned in Marinus's commentary on Euclid's *Data*.[15] Proclus refers to two more works in his commentary on Book I of Euclid's *Elements*; these are: *On the Cochlias* (Περὶ τοῦ κοχλίου)[16] and *On Unordered Irrationals* (Περὶ τῶν ἀτάκτων ἀλόγων).[17] Finally, in Eutocius's commentary on Archimedes' *Measurement of a Circle*[18] we have record of a work by Apollonius called 'Ωκυτόκιον, variously translated as *Quick-Delivery* or *Rapid Hatching*,[19] in which Apollonius supposedly found a closer approximation for the ratio of the circumference of a circle to the diameter. As mentioned earlier, Apollonius was said to have been famed for his work in astronomy. The only solid evidence we have for any Apollonian astronomical work, however, comes from one remark by Ptolemy at the start of Book XII of his *Syntaxis*. Neugebauer has attempted to reconstruct

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[11] *Coll.*, II, Hultsch, pp. 18–24.

[12] In Hogendijk's "Arabic Traces of Lost Works of Apollonius".

[13] *Eucl.*, Heiberg v.6.

[14] Heath, *History of Greek Mathematics*, II,192–193.

[15] *In Eucl. Data*, Heiberg, vi.234.

[16] *In Eucl.*, Friedlein, p. 105.

[17] Friedlein, p. 74. This latter work is also hinted at in one of the scholia to the Tenth Book of the *Elements* (*Eucl.*, Heiberg v.414).

[18] *Archim.*, Heiberg, iii.258.

[19] Heath uses *Quick-Delivery* in his *History*, II,194; Toomer prefers *Rapid Hatching* in his *Apollonius* Conics *Books V to VII*, p. xiii; Hultsch, in his article on Apollonius in *Paulys Real-Encylopädie*, similarly preserves the literal meaning of the word ωκυτόκιον, a quick birth or easy delivery, calling this work by Apollonius *Mittel zur Schnellgeburt.*

this astronomical work by Apollonius in his "Apollonius' Planetary Theory."[20]

That Apollonius's fame was built on the collective effect of all his mathematical work, is hard to doubt. According to Eutocius, however, it was specifically because of Apollonius's establishing that all the conic sections could be obtained from any cone and because of the other wonderful theorems Apollonius proved about conics that Geminus called Apollonius "The Great Geometer."[21] Taking into account that this may only be a judgment of a later time, it still gives *some* basis for believing that, in the earlier period as well, the *Conica* stood out among the rest of Apollonius's works. In any case, "It is clear that Apollonius' treatise was the standard work on conics by the time of Pappus. Probably it had achieved this position long before, but we have no proof of that."[22] At the same time, the general decline in Greek mathematics during the first centuries of the first millennium A.D. is reflected, to some extent, in the decreasing degree to which the *Conica*, while greatly admired, was fully read and understood. In Pappus's time, the complete text of the *Conica* was certainly still being studied since the lemmas in Book VII of Pappus's *Collection* refer to all eight books. Hypatia (370?–415 A.D.), the daughter of Theon of Alexandria, supposedly wrote a commentary on the *Conica*, but there is no way of knowing how much of Apollonius's text her commentary actually covered. Serenus, who probably lived sometime in the middle of the fourth century A.D.,[23] wrote two texts related to conics,[24] *On the Section of a Cylinder* and *On the Section of a Cone.* "In the former, Serenus is at pains to prove that the plane section of a cylinder is an ellipse; the poverty of the latter is apparent from the fact that it is concerned exclusively with plane sections of a cone *through its vertex.*"[25] It is likely, therefore, that already in Hypatia's time the *Conica* was *not* being read in its entirety. What is certain is that by the sixth century, Eutocius of Ascalon

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[20] See also, Neugebauer, "The Equivalence of Eccentric and Epicyclic Motion According to Apollonius."

[21] Eutocius, *Commentaria in Conica*, Heiberg (*Apollonii Pergaei*), II,170.

[22] Toomer, *op. cit.*, p. xvi.

[23] Heath suggests that he came between Pappus and Theon of Alexandria (*History*, II,519).

[24] Besides these two texts, Serenus is also said to have written a commentary on the *Conica.*

[25] Jones, "Later Greek and Byzantine Mathematics," p. 66.

labored to produce an edition and commentary of the first four books only of the *Conica*.[26] About Eutocius's efforts Toomer has said that "The success of Eutocius' version undoubtedly contributed to the disappearance of the Greek original of the last four books of the *Conics*, although this was perhaps inevitable as a result of the narrow scope of interest in mathematics among those concerned with higher education in late antiquity and the Byzantine period. The whole eight books were certainly available to Pappus and to Eutocius, who not only cites Book VI but promises to edit the last four books too if his friend Anthemius showed any interest."[27]

In the Islamic world, as is well known, there was a great effort from the eighth century onward to translate classical texts into Arabic. Our present access to Greek texts, and, therefore, our present understanding of Greek mathematical activity, owes an immense debt to this effort. First of all, Arabic translations remain of Greek texts that have been lost, so that these translations, in many cases, are our *only* access to ancient texts.[28] Second, the translations were made from texts at least as old, and no doubt older, than any Greek texts existing today. Third, unlike writers in late antiquity and Byzantine times, those in the Islamic world who collected, translated, commented on, and built on classical mathematical texts were capable mathematicians who could truly understand and appreciate these ancient works,[29] and, hence, they sought not only the simpler texts, but the most advanced as well.

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[26] The Greek text that we use today of these four books (the *only* books of the *Conica* that have survived in Greek) is based on Eutocius's edition. In the introduction to his commentary on Book III (Heiberg, p. 314), Eutocius attests to there being many older editions of that book. Just how different these other editions were both from one another and from the original *Conica* is difficult to know, and thus the difficulty, also, of knowing how close Eutocius's own edition was to the original. In this connection, one should note that the Banū Mūsā refer to the first four books as Eutocius's restoration, so that at the end of the preface to their edition of the *Conica*, they write, "We have already mentioned that the first four [books] came out according to the restoration of Eutocius, and the three following according to the composition of Apollonius" (in Toomer, *Apollonius Conics Books V to VII*, p. 628). This must be born in mind whenever one takes the text of the *Conica* to be Apollonius's own.

[27] Toomer, *Apollonius Conics Books V to VII*, p. xvii.

[28] For a discussion, see Toomer, "Lost Greek Mathematical Works in Arabic Translation."

[29] Thus, Toomer has said, "The mathematicians who wrote in Arabic (and sometimes in Persian and Turkish) are in a true sense heirs of the Greeks . . ." (*Ibid.*, p. 38).

This great intellectual effort in the world of Islam[30] was cultivated by the ʿAbbasid caliph, al-Mansūr (reigned 754–775), accelerated by al-Mansūr's grandson, Hārūn ar-Rashīd (reigned 786–809), and then, brought to a climax by Hārūn's son, al-Ma'mūn (reigned 813–833), whose "House of Wisdom" (Beit al-hikma) turned Baghdad, in the words of one author, into "a new Alexandria."[31] Among those educated at the "House of Wisdom" in the time of al-Ma'mūn were the three sons of Mūsā b. Shākir, Muhammad, Ahmad and al-Hasan, otherwise known simply as the Banū Mūsā. It is because of the Banū Mūsā that we have a translation of Books V to VII of the *Conica*. The Banū Mūsā acquired first an edition of seven of the eight books, but, ". . . the task [of translating and understanding them] proved impossible for [them] because of the excessive number of errors which had accrued in that treatise . . ."[32] Later, in Syria, Ahmad b. Mūsā found a copy of Eutocius's edition of the first four books of the *Conica* with Eutocius's commentary. This, by the account of the Banū Mūsā, allowed Ahmad to comprehend the text of the *Conica* found earlier, and, in particular, Books V to VII that the earlier text alone contained.[33] Besides the earlier text containing the later books, the significance of this text is that it was evidently pre-Eutocian[34] and, therefore, of an earlier date than any existing text today. The Banū Mūsā did not carry out the translation themselves; it was carried out, under their supervision, by Hilāl b. abī Hilāl al-Himsī (Books I–IV) and by Thābit b. Qurra (Books V–VII). Thābit, incidentally, also translated the works of Archimedes and was probably the main figure in Islamic mathematics in the second half of the eighth century.[35] There are references to another Arabic translation by a certain Ishāq, but little else is known of this.[36]

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[30] This is surveyed well by Lindberg, *The Beginnings of Western Science*, pp. 168–9.

[31] Boyer, *A History of Mathematics*, p. 251.

[32] From the "Preface to the Conics of the Banū Mūsā," in Toomer, *Apollonius Conics Books V to VII*, p. 624.

[33] *Ibid.*, p. 628.

[34] *Ibid.*, p. xviii.

[35] Thābit (836–901) himself was, as Lindberg puts it, ". . . a trilingual pagan (that is, neither a Christian nor a Muslim) . . ." (*op. cit.*, p. 170).

[36] See Toomer, *op. cit.*, p. xviii. Toomer, reasonably enough, identifies Ishāq with Ishāq b. Hunayn, who together with his father, Hunayn b. Ishāq (808–873), the Nestorian Christian head of the House of Wisdom, was behind so much of the prodigious translating activity in the House of Wisdom. Indeed, Ishāq b. Hunayn translated, among other mathematical works, Euclid's *Elements* and Ptolemy's *Almagest* (see Lindberg, pp. 169–170 for more details).

When the *Conica* began to appear in the Latin West is not known precisely, but there is good reason to believe that there were Latin translations of at least parts of the *Conica* as early as the second half of the thirteenth century. The main evidence is that Witelo, in his *Perspectiva*, displayed *detailed* knowledge of propositions from Books I and II of the *Conica* and some knowledge of Eutocius's commentary.[37] Since Witelo knew no Greek and, probably, no Arabic, one must conclude that he had access to a *Latin* text containing a partial translation of the *Conica* already in the 1270's when the *Perspectiva* was written. Unguru suggests that the "author of this hypothetical translation" was Witelo's close friend and collaborator, William of Moerbeke.[38] Serious attempts to produce Latin translations of the *Conica*, however, really began only in the sixteenth century. In Venice, Giovanni-Battista Memmo produced the first translation of Books I to IV in 1537.[39] A better translation was carried out by Francesco Maurolico in 1548, but this was not published until 1654.[40] Maurolico's translation was followed by the most influential translation of the period, that of Federigo Commandino printed in Bologna in 1566.[41] Commandino's edition also included not only the first four books of the *Conica* but also Eutocius's commentary and Serenus's *On the Section of a Cylinder* and *On the Section of a Cone*, and it remained the standard edition of the *Conica* and these related works until the beginning of the eighteenth century.

For the history of Apollonius's text, it is important to note that, throughout the Renaissance and early modern period, the later books

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[37] This is discussed fully in S. Unguru's "A Very Early Acquaintance with Apollonius of Perga's Treatise on Conic Sections in the Latin West."

[38] Ibid., pp. 112, 122. See also, Toomer, *Apollonius Conica Books V to VII*, p. xx.

[39] *Apollonii Pergei philosophi mathematicique excellentissimi Opera per Doctissimum Philosophum Ioannem Baptistam Memum Patritium Venetum Mathematicarumque Artium in Urbe Veneta Lectorem Publicum De Graeco in Latinum Traducta et noviter impressa* (Venice, 1537).

[40] *Francisci Maurolyci Messanensis Emendatio et restitutio Conicorum Apollonii Pergaei. Nunc primum typis excusae, ubi primi quatuor eiusdem Apollonii libri mendis . . . expurgantur, novisque interdum demonstrationibus illustrantur* (Messina, 1654). The relationship between Maurolico's and Memo's editions of the *Conica* has been discussed by K. Saito in his "Francesco Maurolico's edition of the *Conics*."

[41] *Apollonii Pergaei Conicorum libri quattuor. Una cum Pappi Alexandrini lemmatibus, et commentariis Eutocii Ascolanitae. Sereni Antinsensis philosophi libri duo nunc primum in lucem editi. Quae omnia nuper Federicus Commandinus Urbinas mendis quamplurimis expurgata e Graeco conuertit, & commentariis illustrauit* (Bologna, 1566). Commandino also edited Latin translations of Euclid, Archimedes, Aristarchus, Ptolemy, Heron, and Pappus, making him fully deserving of Toomer's compliment that his ". . . services to the restoration of ancient mathematics are rivaled only by Heiberg's" (*op. cit.*, p. xxi).

of the *Conica*, Books V to VIII, were still unavailable (and Book VIII, unfortunately, *remains* unavailable) to mathematically minded readers. The existence of these books was, of course, known from Pappus's account in the *Collection* and Apollonius's own account of them in the preface to Book I, and there were attempts to reconstruct their contents, notable of which was Vincenzo Viviani's reconstruction of Book V in 1659.[42] The Banū Mūsā text, which, as we have seen, contained Books V to VII, was acquired by Jacobus Golius sometime before 1629.[43] Golius, however, kept the text from the mathematics community, reserving for himself the (perhaps deserved) honor of publishing his own translation from the Arabic. In 1667 Golius died; the manuscript, now in the hands of Golius's heirs, was still unpublished. Finally, in 1696, Golius's heirs allowed the Banū Mūsā text and the other manuscripts in Golius's collection to be auctioned in Leiden. From Leiden, the manuscript made its way eventually to Edmund Halley, who, in 1710, produced his edition containing the Greek text of Books I to IV, a Latin translation of Books V to VII, a reconstruction of Book VIII, Pappus's lemmas, and Eutocius's commentary.

The seventeenth century was a remarkably rich period in the history of mathematics; one need only think of such works as Descartes' *La Géométrie* in 1637, Desargues' *Bruillon projet* in 1639 and Pascal's *Essay pour les Coniques* in 1640, Wallis's *Arithmetica infinitorum* in 1655, and, of course, Newton's *Principia* in 1687 to impress upon oneself the extent to which mathematics in that century was progressing in newly defined directions. These works were all written in the interval between Golius's first acquiring the Banū Mūsā text and its publication and translation by Halley, and the significance of that is clear: by the time Halley produced his edition of the Banū Mūsā text in 1710 "its interest was largely historical, whereas had it appeared before, instead of after, the work of the great 17th-century mathematicians such as Descartes, Fermat, Desargues and Newton (all of

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[42] *De Maximis et Minimis. Geometrica Divinatio In Quintum Conicorum Apollonii Pergaei adhuc desideratum* (Florence, 1659).

[43] The story of Golius and the Banū Mūsā text is told in some detail by Toomer (*op. cit.*, pp. xxii–xxv), and by Itard in his "L'angle de contingence chez Borelli," pp. 113–117. Golius's manuscript of the Banū Mūsā text is now in the Bodleian Library, Oxford shelved as Marsh 667, after Narcissus Marsh who sponsored its purchase in Leiden in 1696 and who left it to the Bodleian Library as part of his bequest (see Toomer, *op. cit.*, pp. lxxxv–lxxxvii).

whom were thoroughly familiar with *Conics* I–IV), it could have influenced the development of mathematics in that century."[44] One might consider Halley's text, then, as the beginning of the *modern* history of the *Conica*, that is, the history of the *Conica* as a text less important for what it might teach about conics *per se* as for what it might teach about the Greek *view* of conics.[45]

After Halley's *editio princeps* of the Greek text of the *Conica*, the most significant edition of the first four books of Apollonius's work, as well as Eutocius's commentary on the *Conica* and fragments related to Apollonius's other works, was surely I. L. Heiberg's 1891–93 Teubner edition.[46] This, as Toomer says, ". . . put the Greek text of the *Conics* on a firm scholarly foundation for the first time, based as it was on collations of all the principal mss. and Heiberg's sure philological touch."[47] Heiberg's edition of Books I to IV and Halley's Latin version of Books V to VII are the basis of all the important modern translations of the *Conica*, including Heath's adaptation and

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[44] Toomer, *op. cit.*, p. xxi.

[45] In his biography of Halley, Colin A. Ronan has an entertaining and, on the whole, accurate account of Halley's translating endeavors (*Edmond Halley: Genius in Eclipse*, pp. 186–189), at the start of which Ronan says, "On taking up his Savilian professorship Halley, in company with David Gregory, set to work on the writings of the Greek geometers. The ideas of the Greeks and Romans had interested Halley for some time. In 1696 in an unprinted paper he had discussed and compared their geography with that of his own day and no doubt welcomed this new opportunity which meant that he would come to grips with some early mathematics of considerable importance" (pp. 186–187). From this account, it is fairly clear that Halley was interested in the work of Apollonius and the Greek mathematicians, whom he translated, *as Greeks*, that is, as Greeks *opposed to moderns*. In this connection, it is worth recalling that in the later part of the 17th century there was a vigorous debate as to the relative worth of ancient and modern learning in general) indeed, this is the subject of Swift's 1697 satirical piece, *A Full and True Account of the Battle Fought Last Friday Between the Ancient and the Modern Books in St. James' Library*). We are by no means claiming that Halley's efforts had any polemic overtones, only that they were carried out at a time when ancient science was not considered an obvious source of inspiration for modern science, and that one was, rather, prone to *compare* ancient science with modern science merely for historical interest.

[46] *Apollonii Pergaei quae Graece exstant cum commentariis antiquis.*

[47] Toomer, *op. cit.*, p. xxvii. About Heiberg, Heath writes that he was ". . . a scholar who has earned the undying gratitude of all who are interested in the history of Greek mathematics by successively bringing out a critical text (with Latin translation) of Archimedes, of Euclid's *Elements*, and of all the writings of Apollonius still extant in Greek" (*Apollonius of Perga*, p. x). That gratitude, expressed by Heath in 1896, is, we think it is fair to say, still felt by current historians of Greek mathematics, even though Heiberg's text is not completely free of inadvertences of various kinds.

Taliaferro's English translation, Ver Eecke's French translation, and Czwalina's German translation,[48] although, for the later books, Toomer's edition and English translation of the Banū Mūsā text has, to a great extent, made Halley's translation of Books V–VII obsolete.[49] It is important to note as well that these editions are also the texts on which all traditional[50] *interpretations* of Apollonius's work are based, and, incidentally, the interpretation that we shall present in the present work.

This brings us back to Zeuthen. For of the traditional interpretations based on Heiberg's and Halley's text, by far the most influential as we said above is that set out in Zeuthen's *Die Lehre von den Kegelschnitten im Altertum.* Heath adopted Zeuthen's views almost without question in the long introduction to his *Apollonius of Perga Treatise on Conic Sections* and in the summary of Apollonius's work in volume II of his *A History of Greek Mathematics,* and most other shorter accounts of the *Conica* by other authors stem, one way or another, from these sources.[51]

As for the character and foundation of Zeuthen's interpretation, one must understand that, while Zeuthen was an honest historian who truly wanted to understand the mathematics of the past, he was first of all a mathematician, and he believed deeply that his mathematical training and his knowledge of modern mathematics provided him with the tools necessary to understand ancient mathematics. Zeuthen's great "discovery" in this connection was of "geometric algebra." For him, texts such as the second book of Euclid's *Elements* were only ostensibly about squares and rectangles; they were, in

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[48] These are, respectively, *Apollonius of Perga: Treatise on Conic Sections* (in fact, this is not a proper translation), *On Conic Sections: Books I–III by Apollonius of Perga, Les Coniques d'Apollonius de Perge,* and *Die Kegelschnitte des Apollonios.*

[49] This, naturally, is said with some reservation, for, as Toomer himself admits in the introduction to his *Apollonius Conics Books V to VII,* despite the shortcomings of Halley's version, ". . . every serious student of the *Conics* is greatly in Halley's debt, and I am no exception: in making this edition I have frequently been aided in the understanding of difficult passages by his version" (p. xxvi).

[50] Discoveries and editions especially of Arabic texts might change this (see, for example, Toomer's "Lost Greek Mathematical Works in Arabic Translation" and Hogendijk's "Arabic Traces of Lost Works of Apollonius"), but so far the *basic* interpretive positions have not been significantly altered by the discoveries made up until now; competing views on Books I to IV still look to Heiberg's text for support.

[51] A notable exception is, perhaps, the chapter on Apollonius in Knorr's *The Ancient Tradition of Geometric Problems* in which Knorr sees Apollonius in the context of a general effort among ancient mathematicians to solve certain specific problems.

truth, algebraic texts in content and intention, and geometric only in form—and it was this algebra in geometric form that Zeuthen called "geometric algebra." Armed with "geometric algebra," Zeuthen could read the *Conica* not as a geometrical text, but as an algebraic one; he could analyze it in terms of variables, equations, coordinate systems, and transformations; in short, with "geometric algebra," he could read the *Conica* as he would a modern mathematical work. In recent years, however, historians have realized that "geometric algebra" has little textual basis, is founded on faulty historiographical procedures, and provides no genuine insights into Greek mathematics. In particular, it has produced a hopelessly distorted view of the *Conica*. Given the importance of the *Conica* for our understanding of Greek mathematics, a new and historically more sensitive interpretation of the work, purged of the distorting and foreign influence of "geometric algebra," is, therefore, imperative.

The Plan of the Present Work

The first chapter of Zeuthen's book contains an exposition of "geometric algebra," the first chapter of our book, following the introduction, shows why "geometric algebra" must be rejected. This immediately gives the present book a somewhat polemic tone. This is not our main intention, but it is unavoidable: for although our goal is to provide a reading of the *Conica* stubbornly faithful to the text and aiming only to disclose the idiosyncrasy of Apollonius's thought and that of his time, doing so means supplanting at the same time Zeuthen's deeply rooted interpretation. So, while we do not always explicitly take issue with Zeuthen, he and his followers must ever be, and actually are, in the background.

The second chapter begins the interpretation proper. It takes its name, "The Elements of Conic Sections," from the letter opening Book I of the *Conica*, in which Apollonius calls the first four books "a course in the elements." The chapter, however, focuses mainly on the beginning of Book I, that is, the elements of the elements! It takes a long look at the definitions at the start of Book I and the propositions preliminary to those in which the parabola, hyperbola, ellipse, and opposite sections are first presented and defined. This part of the *Conica* has, in the past, received rather meager treatment, if at all. Yet, it is here that one has the clearest view of Apollonius's

basic assumptions and approach in doing mathematics. For, as we point out, when Apollonius speaks of "elements" he does not mean things easy, but things fundamental. In this chapter too we examine the definitions of the four conic sections themselves, taking particular care to show how and why these definitions are distinct from the principal properties, or *symptomata*, of the conic sections. We look, then, to Books I–III as a whole, without, however, delving exhaustively into their technical details. The discussion is built around three themes: diameters, ordinates, and loci. These themes are central to Books I–III and to all past interpretations of them.

The third chapter is dedicated to the last book in Apollonius's "course in the elements," Book IV. Unlike the other books of elements, Book IV elaborates on a single question: in how many points can two conic sections meet? It is, in this way, an odd book among the elements, and, thus, the main question of this chapter concerns the elementary character of Book IV. Book IV is often presented as tedious and uninteresting, and has, accordingly, merited little attention in the literature. But this is partly so because when its question is framed algebraically it becomes, if not trivial, largely routine. Indeed, in any analytic treatment of curves, the intersection of curves is a matter of solving algebraic equations. For this reason, intersecting curves play a large part in Descartes' *La Géométrie.* To bring out the contrast between an analytic treatment of curves, such as that of Descartes, and a synthetic treatment, as that of Apollonius, we saw it fit to include a short digression concerning *La Géométrie.* Since there is, to our knowledge, no English translation of Book IV, we have also included one as an appendix.

The next three chapters concern the three existing non-elementary books, Books V–VII. Of particular note are our treatments of Books V and VI. These are, respectively, the most and least celebrated of the non-elementary books; indeed, Book V is probably the most celebrated of *all* the books. In our chapter on Book V, we do not question the praise the book has received, but only the reasons for it. Zeuthen and others have lauded the book because they discern in it a treatment of normals to conic sections leading to the equation of the evolute for the parabola, hyperbola, and ellipse. Yet this, we show, has little support in the text and in what Apollonius says about the book, and it is clearly a result of looking at the book as an analytic text. The analytic approach is also the reason for the low estimation of Book VI, for, as an analytic text, Book VI is not very

interesting. Regarded as a purely geometrical synthetic work, however, the book presents the reader with the conceptual difficulty of the equality and, especially, the similarity of conic sections. This difficulty is, no doubt, the reason Apollonius does not include Book VI among "the elements"—a fact little appreciated by those who look at the book from an algebraic point of view. Chapter 5, then, concerns this problem of equality and similarity in Apollonius's study of conic sections. The problem of equality and similarity arises also in the reading of Euclid's *Elements*, and we give some attention to this in the chapter. Chapter 6 looks at two themes related to Book VII. In the first part of the chapter, we discuss the significance of what Apollonius refers to as "diorismic theorems." In the course of this part, we take a brief look also at Halley's reconstruction of Book VIII, which was motivated by his understanding of Book VII. The second part of chapter 6 shows how Apollonius was led by a kind of "analogical thinking" in developing his theme. Here, naturally, we give much attention to the "homologue," defined and used extensively in Book VII.

While chapters 2–6 look at the *Conica* through a microscope, as it were, the next two chapters take a wider view of the book. The first of these continues the discussion of analogical thinking begun in the chapter on Book VII. In particular, we consider the place of the circle in Apollonius's thinking about conic sections: Is the circle just another conic section? Is it a special case of an ellipse, or is an ellipse analogous to a circle? In this chapter, we develop the idea that Apollonius was guided by traditional discussions of the circle, such as Book III of the *Elements*, in his thinking about conic sections. In chapter 8, we consider the sense in which Apollonius saw the *Conica* as one whole book, and we examine the salient structural feature of the work, namely, the division between the elementary and non-elementary books.

The final chapter contains some reflections on the historiography of Greek mathematics. This is an appropriate conclusion since our new interpretation of the *Conica* does not derive from the discovery of new texts or new facts, but only from the application of what has been long recognized as sound historical procedure in general history. Unfortunately, however, what is sound in general history is still not universally practiced, or even preached, among those who purport to study the history of mathematics. We hope, then, that these reflections will leave the reader with a strong sense of how the his-

tory of mathematics ought to be pursued. In addition, together with the book as a whole, this should be seen as discharging an obligation by one of us who, in 1975, spoke of the need to rewrite the history of ancient Greek mathematics. It is only fitting that that obligation be acquitted in an act bringing together teacher and pupil in a common endeavor.

Before beginning our exploration of the *Conica*, one final remark needs to be made. It is a sad fact that the number of people, even among historians of science, who can read Greek is rather small and yet there is no doubt that one who reads a text only in translation is handicapped in his ability to understand it. Still, to read a translation of a text is certainly better than reading an account of a text—all too often an account of a translation of a text! For this reason, in quoting Apollonius and Euclid we decided to use, for the most part, well known translations to which the reader can easily refer. In particular, quotations from Euclid are taken from Heath's translation of the *Elements*, while those from Apollonius are taken from Taliaferro's translation of Books I to III and Toomer's translation of the Banū Mūsā text of Books V to VII. Where there are no translations, of course, we supply our own. Naturally, this is a compromise not lightly made; however, if as a result more readers are encouraged to take up a serious study of the *Conics*, then, we believe, some real benefit will result from this necessary compromise.

CHAPTER ONE

GEOMETRICAL ALGEBRA

A. *Its Formulation*

The Greeks, who were the inventors of mathematics as a postulatory-deductive discipline, did mathematics à la grecque. This is obvious to us. They, however, did not know it. If somebody were able to point out to them this obvious truth, they would have denied it, precisely as Marx denied being a Marxist. The ancient Greeks would have insisted rightly that they were doing mathematics tout court.

The irony of history is that modern mathematicians, in their hypostasis of historians of mathematics as well, fully agree with the ancient Greeks: They too think that Greek mathematics is nothing more than mathematics in Greek, since the truths of mathematics are atemporal and immune to cultural influences. They are universal and eternal. Math is math is math. "The TRUTH: her hallmark is ETERNITY.[1] "But neither thirty years, nor thirty centuries, affect the clearness, or the charm, of Geometrical truths."[2] The import of this position is clear: Greek mathematics is like Egyptian mathematics and both are, when push comes to shove, like Babylonian, Islamic, Chinese, or Mayan mathematics. There is just one universal, all-embracing, cosmic mathematics, which makes Greek mathematics *ours* as well. Hence, there is nothing to prevent us from understanding Greek mathematics while using all the mathematical paraphernalia at our disposal to remove "the trivial and the accidental", having to do with the idiosyncratic Greek way of doing things, in order to cut through to what is, after all, pure, unadulterated, eternally valid mathematics.

It is precisely this approach that engendered Geometrical Algebra

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[1] "Die Wahrheit, sie besteht in Ewigkeit "reads the original of A. v. Chamisso; the English translation is by Max Delbrück in *The Mathematical Intelligencer* 10 (1988), p. 53.

[2] C. L. Dodgson, as quoted in Benno Artmann, *Euclid—The Creation of Mathematics* (Springer, 1999), p. 45.

and insured its success and wide adoption. What, then, is Geometrical Algebra? And how did the development of mathematics make such a reading of Greek mathematical texts possible? In other words, how did Geometrical Algebra come into being? To answer these questions, we need first to say something about the nature of Greek mathematics.

According to Eudemus's lost history of geometry, as reported in Proclus's *Commentary on the First Book of Euclid's Elements*,[3] geometry originated in Egypt as land measurement, while arithmetic began with the Phoenicians as calculation, in each case the generating cause being practical necessity. The myth, as perpetrated by the Greeks themselves, has it that the Greeks learned geometry from the Egyptians. But Greek and Egyptian geometry are worlds apart. Whatever the Greeks may have learned from the Egyptians, they transfigured their putative inheritance, lifting it from the domain of mensuration and flood repair into the realm of an ideal, precise, and rigorous reality. After an early (and historically problematic) beginning with Thales and the Pythagoreans,[4] Greek mathematics grew explosively[5] and, by the fifth century B.C., succeeded in discovering, *i.e.*, proving, incommensurability, an extraordinary mathematical feat, well beyond the highest achievements of pre-Hellenic mathematics. The discovery of incommensurability shattered the innocence of the Pythagorean sect's (school's) naive belief that all segments were commensurable, or, which is the same, that all is number. It led to Eudoxus's reformulation of the theory of proportions on purely geometrical grounds and to the thorough separation of geometry from arithmetic. To a large extent, Greek mathematics became Greek geometry. Numerical considerations became foreign to geometrical arguments, any attempt at mixing the two falling under sinful rubric of *metabasis*.

The change mathematics underwent after the collapse of the view that all was number meant that geometry, being an independent

[3] Cf. Morrow's translation (Princeton, 1992), pp. 51ff.

[4] Reliable information on Thales is scarce,and that on Pythagoras the mathematician discredited after Burkert's book (*Weisheit und Wissenschaft*, 1962), translated into English ten years later as *Lore and Science in Ancient Pythagoreanism*. Cf., however, Leonid Zhmud, *Wissenschaft, Philosophie und Religion im frühen Pythagoreismus* (Berlin: Akademie Verlag, 1997) for a valiant attempt at a defense of the reliability of ancient sources on Pythagoras.

[5] Reviel Netz, *The Shaping of Deduction in Greek Mathematics* (Cambridge, 1999), pp. 272–75.

mathematical discipline, was not reducible to arithmetic (or to anything else, for that matter) and, necessarily, that elucidation of geometrical issues by non-geometrical means was impermissible, as it missed entirely the point by mixing unmixables and sinning thus against the explicit prohibition of metabasis.[6] This should also mean *manifeste ab initio* that modern interpreters of Greek geometry, who gain hermeneutic aid and comfort from invoking non-geometrical tools in their interpretations, transgress sinfully against the historical character of their object of inquiry, providing, at best, anachronistic explanations. To this one should also add the trivial fact that algebra could not be mixed with geometry, or with anything else, for that matter, since algebra did not (and could not) exist.

This brings us naturally to Geometrical Algebra. What exactly is it? To put it bluntly, but accurately, it is the view that Greek geometry, at least in its fundamental, operational aspects, is nothing but algebra in disguise. Whenever *we* feel, with our mathematical instincts, that something looks awkward geometrically, despite its clear-cut outward appearance, it is because it is algebra in disguise. Why does algebra need the geometrical garb? This is not at all clear and the various attempts at an answer strike us as incoherent on various levels. They amount to the view that, after the discovery of the "irrational" by the Pythagoreans, algebra could appear only in geometric apparel, for the sake of rigor. We are not told where Greek algebra was before that fateful discovery, nor is the secret divulged to the curious student under what attire, if any, it was hiding then, in pre-lapsarian days, nor, needless to say, are we informed adequately about the narrowly specific reasons for the change of toilet and for the specific garments interchanged.

This is a curious methodological situation. When unpacked it reveals the following startling content: "The reasoning of Greek mathematics, the line of attack of its various problems, the solutions provided to those problems, *etc.* all are essentially *algebraic*, though ... for reasons that have never been fully elaborated, attired in geometrical accouterments. We are, then, not only authorized to look

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[6] Aristotle, *Post. An.* 75a38–75b6; it is here that Aristotle says: "It follows that we cannot in demonstrating pass from one genus to another. We cannot, for instance, prove geometrical truths by arithmetic. . . . in the case of two different genera such as arithmetic and geometry you cannot apply arithmetical demonstration to the properties of magnitudes unless the magnitudes in question are numbers." (This is G. R. G. Mure's tanslation.)

for the algebraic "subtext" (so to speak) of any geometrical proof, but it is indeed wise (historically!) always to transcribe the geometrical content of any proposition in the symbolic language of modern algebra, especially when the former is particularly cumbersome and awkward, while the recourse to the latter always makes the logical structure of the proof clear and convincing, without thereby losing anything not only in generality but also in any possible *sui generis* features of the ancient way of doing things.

"In other words, there is nothing unique and (ontologically) idiosyncratic concerning the way in which ancient Greek mathematicians went about their proofs, which might be lost in the process of translation from the geometrical to the algebraic language; the main reason for this being that the ancient mathematical reasonings and structures *are* indeed substantially algebraic."[7]

B.

1. *Its Origins*

When did geometrical algebra come into being, *i.e.*, how did the growth of mathematics enable the algebraic reading of Greek geometry? Its exact beginnings are difficult to pinpoint with adequate precision, but it is both easy and accurate to state that its *terminus a quo* falls squarely after the beginning of our era, in the period of decline of Greek mathematics. This is geometrical algebra *avant la lettre*, recognizable only in hindsight, and its perpetrator is Heron of Alexandria, at the end of the first century A.D., in his lost Commentary on Euclid's *Elements*, preserved fragmentarily in an-Nairizi and Proclus. It is there that Heron sketches alternative, *linear*, non-diagrammatic proofs to propositions II.2–10 in the *Elements*, based on II.1, which is preserved in its geometrical Euclidean figurative appearance. Algebraically, II.1 states $a\ (b + c + d + \ldots) = ab + ac + ad + \ldots$, while its Euclidean enunciation reads: "If there be two straight lines, and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles con-

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[7] S. Unguru, "On the Need to Rewrite the History of Greek Mathematics", *Archive for History of Exact Sciences*, vol. 15, No. 1 (1975), pp. 67–114, at 69–71. Appropriate references substantiating this characterization appear in the cited article.

tained by the uncut straight line and each of the segments". Though mathematically equivalent, historically and epistemologically the two enunciations are poles apart.[8] Be that as it may, however, Heron uses II.1, which he considers irreducible to non-figurative appearance, to prove quasi-algebraically as it were the following nine propositions. As an example, we can use II.4, which in Euclid reads: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments."

The proof is quite lengthy and rigorously geometrical, *i.e.*, it is constructive and manipulatory at the diagrammatic level, and it relies exclusively for its conclusions on propositions proved in Book I, not on II.1. For Heron, on the other hand, all that is needed is a mere line and the proof is an immediate consequence of II.1! Thus, if ABC is a straight line with B lying between A and C, $\underline{A \qquad B \;\; C}$, and if $AB = a$ and $BC = b$, then a repeated application of II.1 to the line ABC (as it were undivided) and to the segments AB, BC of the "other" cut line, ABC, leads (in anachronistic, symbolic writing) to the following immediate conclusion: $(a + b)(a + b) = (a + b)a + (a + b)b = a^2 + ba + ab + b^2 = a^2 + b^2 + 2ab$.[9] This is most certainly *not* what Euclid does, but the buds of geometrical algebra as they appear in this "innocent" Heronian procedure will blossom in due time into an alluring, widely scattered, and poisonous historiographic flower.

There are no genuine instances for the use of geometrical algebra in either pre-Euclidean or Euclidean times. The scholia to the first five propositions of Book XIII of the *Elements*, which crept into the manuscript tradition some time before Theon's recension, certainly do not qualify: First, they are late interpolations, displaying the same *linear* character as that of Heron's alternative proofs; next, they are spurious, being ". . . altogether alien from the plan and manner

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[8] Unguru, *ibid.*, passim; cf. also, S. Unguru and David E. Rowe, "Does the Quadratic Equation Have Greek Roots? A Study of 'Geometric Algebra', 'Application of Areas', and Related Problems," *Libertas Mathematica*, vol. I (1981), pp. 1–49, and vol. II (1982), pp. 1–62; S. Unguru, "History of Ancient Mathematics: Some Reflections on the State of the Art," *Isis*, vol. 70 (1979), pp. 555–565. These articles also contain references to the various published reactions to the "On the Need to Rewrite"-article.

[9] B. L. van der Waerden, *Geometry and Algebra in Ancient Civilizations* (Berlin, etc.: Springer-Verlag, 1983), pp. 182–83.

of the *Elements*,"[10] thirdly, and most significantly, the method of proof they display is totally foreign to classical Greek mathematics. "There is not one shred of reliable historical evidence to support the speculations of Bretschneider, Heiberg, etc. that this method represents, [in Heath's words] '. . . a relic of analytical investigations by Theaetetus or Eudoxus . . .'; indeed the whole history of Greek mathematics seems to exclude such an inference. But even if one believes Heiberg's later dating (in 'Paralipomena zu Euklid', *Hermes, 38* (1903), 46–74, 161–201, 321–356), namely that the author of these interpolations is Heron of Alexandria, this would still make these additions some 400 years younger than Euclid and would place them comfortably (to the exclusion of Pappus and Diophantus) *after* the decline of classical Greek mathematics."[11]

Nor are the great post-Euclidean mathematicians of the third century B.C., Apollonius and Archimedes, the initiators of an ". . . increased use of an informal, but subtle and penetrating, algebra of line lengths . . ."[12] The examples brought up to support the existence of this "algebra of line lengths" are implausible and, therefore, questionable, since, though they may indeed point to a freer manipulation of line lengths, they hardly represent an "algebra of line lengths" Thus, proposition II.1 of *On the Sphere and the Cylinder* is actually proven geometrically, in the best tradition of Euclid's *Elements*.[13] The same is true about II.4 of the same work: the proof is Euclidean, though there are, clearly, differences, some of which do point to a freer manipulation of lines. Still, and this is crucial, in both cases the line lengths *are closely linked to two or three-dimensional figures*. It is difficult to see how somebody whose mind is 'uncorrupted' by algebraic reasoning and manipulations could describe Archimedes' proofs as "algebra-like manipulations of line lengths", though this inappropriate label is less offensive than "geometric algebra".

The next significant stage in the development of Geometrical Algebra is provided by Arabic mathematics, which is the cultural heir of Greek mathematics, to which it added the arithmetic and

[10] T. L. Heath, *The Thirteen Books of Euclid's Elements*, vol. 3 (New York: Dover, 1956), p. 442.

[11] Unguru, "On the Need" (n. 7, *supra*), p. 107.

[12] Michael S. Mahoney, "Another Look at Greek Geometrical Analysis", *Archive for History of Exact Sciences*, vol. 5 (1968), pp. 318–48, at 337.

[13] J. L. Heiberg, ed., *Archimedes Opera Omnia*, 2nd ed. (Leipzig: Teubner, 1910), vol. 1, pp. 170–74.

algebra of the Hindus. Al Khwarizmi's *Hisab al-jabr w'al-muqabala*,[14] one of the main sources for the introduction of algebra in the West, deals systematically with the solution of linear and quadratic equations. This is rhetorical algebra at its best, innocent of any algebraic symbolism whatever and lacking even the abbreviatory mechanisms of Diophantus's *Arithmetica.* The treatise is divided in two parts, the first giving numerical recipes for solving the different types of equations (whose existence is a result of the fact that Khwarizmi does not recognize negative coefficients), without any justification, while the second part vindicates the recipes *geometrically.* And this vindication is highly interesting for our purposes, since it exploits the *Elements* algebraically, using for the first time, perhaps, manifestly and fully propositions from Book II of Euclid's *Elements*, one of the mainstays of the geometrical algebraists, for solving equations and for supplying a rationale for their solution procedure.

Here is an example. One type of quadratics is, in Khwarizmi's jargon, *Squares* (mal) and *Roots* (jadr) equal *Numbers* (ʿadad), by which is meant what *we* would write as $ax^2 + bx = c$, though, of course, Khwarizmi expressed each type by means of specific numerical examples, before he gave its solution recipe. In this case, the particular example is "Squares and ten Roots equal 39 Numbers", and the recipe solution as it appears in the first part of *Al-jabr w'al-muqabala* is as follows: Take half the roots, 5, multiply the result by itself, obtaining 25, add this result to the numbers, 39, obtaining 64, take its square root, or 8, and subtract from it half the roots, 5, and there remains 3. This is the searched for root.

In the second, justificatory, part of the treatise, a part lacking, as we know, from run-of-the-mill books of recipes, like cookbooks, say, the rationale for the effectiveness of the recipe solution is given in the following diagrammatic terms:

If AB is the root (see Fig. 1), the square on AB represents the square appearing in the equation. Prolonging both AB and BC to I and E respectively, so that $AI = CE = 5$, half of the roots, and completing the square on BI and BE, *i.e.*, square $BEGI$, one observes that the areas added to the square on AB are twice five roots (rectangle AH = rectangle $CF = 5x$, $AB = x$), *i.e.*, ten roots, making together

[14] There is an English translation of the Latin version of the treatise in: L. C. Karpinski, *Robert of Chester's Latin Translation of the Algebra of al-Khowarizmi* (New York: Macmillan, 1915).

39 numbers (in modern notation, $x^2 + 10x = 39$), and the square *HF*, whose area is 25 numbers. To obtain the area of the big square, *BEGI*, one must add 25 to 39, which is 64.

Extracting the square root from 64, one gets the side of the square, which is 8, and subtracting 5 from 8, one obtains 3 as the value of the unknown root.

This is clearly a *metrical* geometrical argument, grounded on II.4 *Elements*, and advanced specifically for the sake of justifying the mechanical, recipe-like solution of an algebraic equation. Both solution and justification amount, as obvious, to "completing the square".[15] Still, it is not geometry for its own sake that we face, but rather geometry for the sake of algebra. And it is impure geometry at that, since, unlike Greek geometry, metrical considerations have crept in, for ulterior motives, so to speak, namely, to vindicate a purely algebraic procedure for solving quadratics of a certain type. The geometry is ancillary, derivative, subjugated by algebra whose vassal it is, though, interestingly enough, it alone is thought to be able to vindicate rationally the Senior Feudal's steps and measures. This can, then, rightfully be seen as a certain brand of geometrical algebra and be designated as such, unlike those parts of Greek geometry (Book II and parts of Books VI and X of the *Elements*) wrongly baptized with the same name, which they carry illegitimately, since they embody geometrical truths for their own sake.

There are other genuine instances of geometric algebra in Islamic mathematics. We cannot survey them all here, but we shall mention the names of Thabit ibn-Qurra, Abu Kamil, and Omar Khayyam. Abu Kamil, in his *Algebra*,[16] which is substantially a commentary on and an enlargement of Al Khwarizmi's treatise, pursues the exploitation of Greek geometry for algebraic means. An example should suffice. It is that of "square and numbers which are equal to the roots," specifically, "the square and 21 are [equal] to 10 roots",[17]

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[15] There is another variation on "completing the square" in the *Hisab al-jabr*, in which the roots are divided into four parts, and four rectangles, each equaling $2.5x$, are arranged around the four sides of the square x^2, and the big square completed thereafter. Though the diagram is different (cf. Edward Grant, ed., *A Source Book in Medieval Science* (Cambridge, Mass.: Harvard Univ. Press, 1974, p. 110), the reasoning is unaffected.

[16] *Kitab fi al-jabr wa'l-muqabala*, edited and translated by Martin Levey, *The* Algebra *of Abu Kamil* (Madison: Univ. of Wisconsin Press, 1966).

[17] *Ibid.*, p. 38.

which appears also in Khwarizmi.[18] The solution recipe for x[19] is as follows: "... one takes 1/2 the roots, *i.e.* 5. Multiply it by itself; it is 25. Subtract 21 from it. There remains 4. You take its root; it is 2. Subtract it from 1/2 the roots, 5, and 3 remains. It is the root of the square; the square is 9."[20] The geometrical justification comes immediately thereafter and, again, it exploits Book II of the *Elements*, this time proposition II.5, to vindicate the mechanical recipe given before.

The enunciation of II.5 is: "If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half."[21] It will be easy for the attentive reader to note Abu Kamil's use (for a Greek, this is clearly a *misuse*) of Euclidean geometry for algebraic purposes (the following diagram is essentially Euclid's diagram in II.5):

Take the square number *AB* (see Fig. 2), add to it 21, represented by the rectangle *CB*; the area *AD* is equal, then, to 10 roots, which means that *AC* or *KD* must be equal to 10 roots. Bisect *KD* at *E* (this brings the diagrammatic situation to the state of II.5, *i.e.*, rect. (*CF*,AF) + sq. on FL = sq. on LC) and draw the perpendicular *EL* to *KD*, extending it beyond *E* to *M*, so that *LM* = *LC*, and complete the squares *LMNC* and *MOPE*. The areas *LB* and *OD* are equal, as complements in a parallelogram (by I.43) and square *LN* = *25*. Now the gnomon *LEPONC* = *21* = rectangle *BC*. Hence, the square *MP* = 4 and its side, *EP* = 2. But *EM* = *EB* and *KE* = *5*, therefore the unknown root *KB* = *AF* = *5* − *2* = *3*.

Abu Kamil is aware that the equation he deals with in this case has two positive roots and, by means of another, but related, diagram, he finds the second root as equal to 5 + 2 = 7, justifying again geometrically a recipe-like procedure, given beforehand.

These examples should suffice as evidence of what is genuine geometric algebra, *i.e.*, algebra which uses geometry as its handmaid for clearly algebraic purposes, specifically in the Islamic case, in order to justify rationally a mechanical algebraic rule for the solution of

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18 Symbolically, this is, of course, $x^2 + 21 = 10x$.

19 Unlike Khwarizmi, Abu Kamil gives also separate solutions for x^2.

20 *Ibid.*

21 T. L. Heath (n. 10 *supra*), vol. 1, p. 382.

quadratics. But, before we leave the Muslims, a few words are in order about the Persian Omar Khayyam's work in algebra. In a treatise of genius rich in novelty, *Algebra*,[22] Omar Khayyam transcends the preoccupations of most of his predecessors with quadratics and adds to his interests the solution of cubics, to which he denies arithmetical solutions, however, declaring that their only available solutions are geometrical. These solutions he provides by means of the intersections of conic sections, a method of Greek origin, used, however, by the Greeks (Menaechmus and Archimedes) exclusively for the solution of classical geometric problems.[23] The most remarkable thing, perhaps, about Omar Khayyam's methodology is his denial of any essential disciplinary differences between geometry and algebra, in which he sees the same field of inquiry in two different guises: "Whoever thinks algebra is a trick in obtaining unknowns has thought in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras are geometric facts which are proved."[24] Only under such a conception, we feel, is it legitimate to speak of geometrical algebra. This conception, legitimizing, perhaps, talk of an Islamic geometrical algebra, is, needless to say, foreign to the Greek mathematician.

The next important historical stage in the development of geometrical algebra occurs in the sixteenth and seventeenth centuries. Indeed, it is a feature of modern mathematics that its creators think (practically all of them) that an entirely adequate way to understand ancient Greek mathematical texts is through their algebraic reading. This opinion is coupled with their claim to have unveiled the secret of ancient mathematics in that *mathesis universalis*, algebra, a claim derived, in part, from the high estimation of their own originality.[25] Their views are foreshadowed in the sixteenth century by Petrus Ramus, logician and rhetorician, who seems to be the first to "dis-

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[22] D. S. Kasir, ed., *The Algebra of Omar Khayyam* (New York: Columbia Teachers College, 1931. This is a published Ph.D. dissertation.

[23] In Archimedes' case, for instance, the problem is "to cut a given sphere, so that the segments of the sphere shall have, one towards the other, a given ratio." This is prop. 4 of the second book of *On the Sphere and Cylinder* (Heiberg's edition, vol. I, 186.15–192.6).

[24] A. R. Amir-Moez, "A paper of Omar Khayyam," *Scripta Mathematica*, vol. 26 (1963), pp. 323–37, at 329.

[25] David R. Lachterman, *The Ethics of Geometry A Genealogy of Modernity* (Routledge: New York & London, 1989), *passim*.

cern" that the algebraic art must underlie some parts of Euclid's *Elements* (Books II and VI) and, perhaps, also Greek analysis.[26]

The coryphaei of the new mathematics, its true founders, Viète, Descartes, and Fermat, followed Ramus in his belief that algebra lies at the root of Greek analysis. William Oughtred in the seventeenth century, in his most famous mathematical textbook, *Clavis mathematicae*, is also of the opinion that algebra can serve as a means of understanding difficult problems in Euclid, Archimedes, Apollonius, and Diophantus.[27] It is clear, then, that our nineteenth and twentieth century historians of mathematics can boast of a lengthy and aristocratic mathematical lineage. They have made Oughtred's "insight" the keystone of their methodological and interpretive approach!

What we would like to do now is use the relatively unknown *Algebra Geometrica* of Paolo Bonasoni as one peculiar example of what is involved in reading the *Elements* algebraically.[28]

Clearly, the *sine qua non* condition for such an undertaking is the existence of an algebraic disciplinary matrix ready for geometrical manipulation in dealing with the *Elements*; this also means that the geometrical manipulator of algebra is fully aware of the different disciplinary identities of the two provinces of knowledge he combines to get answers to his questions. That this is indeed the case is obvious in the *Algebra Geometrica*:

> Algebra was discovered by the ancients in connection with Arithmetic, and it is the most noble, and by no means sufficiently celebrated technique of numbers.... But when I thought out a similar art for Geometry not long ago, and found it, when dealing with magnitudes, to be of a supreme subtlety and utility no less admirable than that of the other art in numerical matters, I decided, . . . to bring forth into the light a thing hidden till now . . .[29]

And:

> . . . I am of the opinion that this Geometrical Algebra must be judged a more outstanding candidate for that name, because Numerical Algebra

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[26] Michael S. Mahoney, "Die Anfänge der algebraischen Denkweise im 17. Jahrhundert," *Rete*, vol. 1 (1971), pp. 15–31, at 25.

[27] *Ibid.*, p. 28, n. 49.

[28] Robert Schmidt, ed. and transl., *The* Algebra Geometrica *of Paolo Bonasoni circa 1575* (Annapolis, MD: Golden Hind Press, 1985); the book appears, then, in the midst of a period of intense algebraic activity!

[29] *Ibid.*, p. 2.

> most often exhibits the quotiety in surd numbers, which cannot actually be given in the nature of things,while Geometrical Algebra lacks so great and continual a disadvantage, and where it can be applied always exhibits perceptible and comprehensible solutions, by which both intellect and sense may be brilliantly satisfied, as will be clear in what follows.[30]

Bonasoni, who boasts about being the inventor of geometrical algebra, has written a brief treatise in three books, in which the interpenetration of algebra and geometry is clear and distinct. We shall give a few examples.

The First Theorem of the First Book says that when two lines are given and one of them is divided into any two parts, then the rectangle formed by the two given lines, less the rectangle formed by the uncut line and a segment of the divided line, equals the rectangle formed by the uncut line and the other segment of the divided line. In other words (Fig. 3),

> if AB and DE are the given lines and AB is divided arbitrarily at C, then
>
> $$AB*DE-DE*AC = DE*CB.$$

By II.1 *Elements*, $AB*DE = AC*DE + CB*DE$, and by transferring $AC*DE$ to the first member of the equality (which seems, algebraically, to be the whole point of the theorem), one obtains $AB*DE-AC*DE = CB*DE$, q.e.d.[31]

The second theorem also involves the transfer of one expression from side to side by changing its sign.[32] We shall now give examples of Problems and Questions as they appear in the three books, in order to display the relative ease and matter of factness with which elementary algebraic operations are performed by Paolo Bonasoni in geometrical contexts, something, needless to say, the Greeks never did.

The First Problem of the First Book is: "*To find a straight line whose square, together with the rectangle that is made under the line itself and another given straight line, is equal to a given rectilinear figure.*[33] Thus (Fig. 4), if AB and C are the given line and rectilinear figure respectively, we must find another line whose square together with the rectangle formed by it and AB equals C.

[30] *Ibid.*, p. 3.
[31] *Ibid.*, pp. 5–6.
[32] *Ibid.*, p. 6.
[33] *Ibid.*, p. 7.

Let $AD = DB$ and let AB be produced to F in such a manner that $(DF)^2 = (DA)^2 + C$.[34] The square equivalent to C, or, to use Bonasoni's words, "... a line equaling in square the rectilinear figure C",[35] is found easily by means of II.14 *Elements*. The claim is, then, that

> $(AF)^2 + (AF)*(AB) = C$, in other words, that the sought after line is AF. There are two versions of the proof. Here is the first:
> By II.6 *Elements*, $(FB)*(AF) + (AD)^2 = (FD)^2$. But, by the previous construction,
> $(FD)^2 = (AD)^2 + C$. $\therefore (FB)*(AF) + (AD)^2 = (AD)^2 + C$. $\therefore (FB)*(AF) = C$.
> But, by II.3 *Elements*, $(FB)*(FA) = (AB)*(FA) + (FA)^2$. $\therefore (FA)^2 + (AB)*(FA) = C$, q.e.d.

The alternative proof ("aliter") is immediate and relies on V.29 *Elements*:

> If a parallelogram FG (Fig. 5), equal to rectilinear figure C, is applied to line AB, such that it exceeds it by square FE, then, since, by construction, $FG = C = FE + AG$, *i.e.*, $(FA)^2 + (FA)*(AB) = C$, q.e.d.

It is easy to detect the algebraic veil spread all over the proof and marking clearly its main contours. But the algebraic motivation and structure of the third problem of Book One, the last in the book, is even more manifest. In it elementary algebraic manipulations leap to the eye of the alert reader, as the requisite tool for the solution of the problem.

The Third Problem reads: "To find a straight line whose square, together with a given rectilinear figure, is equal to the rectangle made under the line itself and another given straight line. It is necessary, however, that the given rectilinear figure is not greater than the square of half the given straight line."[36]

Here is the proof, where B is the given figure and CD the given line (Fig. 6):

> Let $CF = FD$. (1) If $(CF)^2 = B$, then CF is the sought line, since $(CF)^2 + B = (CF)*(CD)$; all the last equality says is $2(CF)^2 = 2(CF)^2$.
> (2) However, if the sought line is smaller than CF, then let GF be taken such that

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[34] Bonasoni's text being a consciously algebraic text, though avowedly in mixed geometrical garb, the use of this symbolism to abbreviate Paolo's rhetorical description is justified. When Paolo says "the square of DF" and we say $(DF)^2$, this is truly an abbreviation. This applies to all our abbreviations.

[35] "... linea ... quidem est potens rectilineum C, ..." (p. 110).

[36] *Ibid.*, pp. 13–14.

♠ $(GF)^2 = (CF)^2 - B$ and let $GF = FH$. The claim is that a twofold answer will be possible, namely CG and GD.

First CG. Since $(CF)^2 = (CG)^2 + (GF)^2 + 2(CG)*(GF)$, while, by ♠, $(GF)^2 + B = (CF)^2$,
$\therefore (CG)^2 + (GF)^2 + 2(CG)*(GF) = (GF)^2 + B$
$\therefore (CG)^2 + 2(CG)*(GF) = B$ ♦
$\therefore (CG)^2 + (CG)^2 + 2(CG)*(GF) = B + (CG)^2$
$\therefore 2(CG)^2 + 2(CG)*(GF) = (CG)^2 + B$. But, the first member of this last equality equals
$2(CG)*(CG + GF) = 2(CG)*(CF) = (CG)*2(CF) = (CG)*CD$.
$\therefore (CG)*(CD) = (CG)^2 + B$, q.e.d.

Next GD. By ♦, $(CG)^2 + 2(CG)*(GF) = B$. But,
$(CG)^2 + 2(CG)*(GF) = (CG)*(CG + GF + GF) = (CG)*(CF + GF) = (CG)*(CF + FH) =$
$= (CG)*(CH) = (CG)*(GD)$
$\therefore (CG)*(GD) = B$
$\therefore (CG)*(GD) + (GD)^2 = B + (GD)^2$
$\therefore (GD)*(CD) = B + (GD)^2$, q.e.d.

There should be no doubt about the telling algebraic character of the proof, involving as it does simple arithmetico-algebraic operations enacted, it is true, on linear, geometric entities. Bonasoni's lengthy treatment of the problem, spread over almost seven pages,[37] contains also an alternative proof, based on VI.28, *Elements*, which enables him to point to the Euclidean *diorismos* as an explanation of the one appearing in the enunciation of the problem, and to emphasize proudly his primacy in discovering that VI.28 too has two solutions! This is indeed a telling point, since solving an algebraic equation means finding all possible solutions, while a geometrical construction is done when the task set out is completed, so that one solution is typically enough.

In the second book of the treatise, algebra enters through the main door:

> In questions that are soluble by this art, a straight line is set out in any manner whatsoever denoting the line to be investigated. For convenience we will designate it thus: A^1. Moreover, one must reason with this line according to what is proposed in the question (just as one must usually work with the *res* in Arithmetical Algebra), until an equation is found in which there is either had another line equal to the line A^1 itself, or one of the three problems above appears.[38] We

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[37] *Ibid.*, pp. 13–22; pp. 117–125 of the Latin text.
[38] The three problems appearing in Book I are meant.

call the former equation simple and the latter ones composite. In the simple equation the thing we are intent upon is had, of course, from the line which is found equal to A^1 itself, while in a composite equation it is had by the solutions of those same three problems.[39]

Some illustrations of "the art" in action are in order. In them the dictates algebra imposes on geometry are conspicuous. The Second Question reads: "To divide a straight line *DB* (Fig. 7) so that the rectangle under the whole and the lesser part, together with a rectilinear figure *C*, is equal to the rectangle under the whole and the other part."[40] The solution follows:

If *A*^1 is the lesser part, then *DB* − *A*^1 is the greater. But, *DB**(*DB* − *A*^1) = $(DB)^2$–*DB***A*^1.
∴ *DB***A*^1 + *C* = $(DB)^2$ − *DB***A*^1. Now one adds the rectangle *DB***A*^1 to both sides of the former equality and one obtains, 2*DB***A*^1 + *C* = $(DB)^2$ and by subtracting *C* from both members of the same equality, one gets, 2*DB***A*^1 = $(DB)^2$ − *C*. ∴ *DB***A*^1 = (DB^2 − *C*) − *DB***A*^1. If one makes $(DB)^2 - C = E^2$, then 2(*DB*)**A*^1 = E^2. ∴*A*^1*(2*DB*) = E^2 (this is a typical algebraic ploy). Now, if one takes *F* such that 2*DB***F* = E^2, then, assuming *DG* = *F*, *A*^1*(2*DB*) = *F**(2*DB*), hence, *A*^1 = *F*. It should be pointed out here that this way of solving the question is typical for "putting a geometrical problem into equation": (1) One selects the unknown, *A*^1; (2) one finds a numerical solution, in the case at hand, the line *F*; (3) the numerical solution is given a geometrical interpretation, in our case, *DG* is a segment of *DB* equal to *F*.

There follows a construction, which we omit, in which *DG* is taken equal to *F*, and the claim is made that this solves the problem. Then follows the demonstration:

Since, by construction, *DG**(2*DB*) = E^2, ∴*DG**(2*DB*) + *C* = E^2 + *C*, ∴2*DG***DB* + *C* = DB^2, ∴*DG***DB* + *C* = DB^2–*DG***DB*. But, DB^2–*DG***DB* = *GB***DB*, ∴*DG***DB* + *C* = *GB***DB*, q.e.d.

The algebraic way of thinking literally jumps at the reader, though Bonasoni's marginal citations refer to the *Elements* as warranting the various algebraic inferences during the discussion, construction, and demonstration. This is an entirely appropriate statement: Bonasoni is doing algebra because he says he does, because of his interest in all possible solutions to a problem, because there is a conscious

[39] *Ibid.*, p. 21.
[40] *Ibid.*, pp. 23–24.

parallelism between his arithmetical and geometrical manipulations, because of the serial nexus we witnessed above, involving the chain Unknown-Number-Geometrical Application, and, finally, because of the standard way of organization of his problems.

The same is true of the Third Question of Book Two; in this case, however, we shall point out explicitly the appeal to Euclidean propositions during its treatment.

The question reads:

> Given two straight lines *BF* and *D* [see Fig. 8], to find a third line less than *BF* that makes with *BF* a rectangle equal to the rectangle made under *D*, and the excess by which *BF* surpasses the third line itself.[41]

We would call the third, unknown, line x and "put the problem into equation" as follows:

> $BF*x = D*(BF - x)$, from which one obtains, $x\ (BF + D) = BF*D$ and, $x = (BF*D)/BF + D$.

Bonasoni does essentially the same in longer compass and paying lip tribute to Euclid.

> If A* is the unknown third line,[42] then $BF*A* = D*(BF - A*) = D*BF - D*A*$.

$\therefore$ $BF*A* + D*A* = D*BF$; hence, by II.1 *Elements*, $\therefore$ $A*(BF + D) = BF*D$. Now, if a line C were taken, such that

$C*(BF + D) = BF*D$, then $C = A*$ (which is A^1 in the diagram). Here Bonasoni justifies his previous steps by the marginal reference to what he calls a corollary to I.45 *Elements*.[43] In the Construction, it is required to find C such that (1) $C*(BF + D) = BF*D$. It follows that $C < BF$, because $(BF + D)/D = BF/C$.

Now, if BG is taken equal to C, then $GF = BF - C$, and he claims, referring to either VI.14 or 16, that $C*BF = GF*D$. This he shows in the Demonstration as follows:

> Since $C*(BF + D) = BF*D$, by (1) above, it follows that
> $C*BF = BF*D - C*D = D*(BF - C) = D*GF$, q.e.d.

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[41] *Ibid.*, p. 26.

[42] For practical reasons, we adopt this simpler notation.

[43] Bonasoni's marginalia to Euclid refer to the edition of Clavius, which differs in its numbering of propositions from ours.

Once more, the algebraic approach is prevalent.

The last question of Book Two is the tenth, which involves more algebraic sophistication than the previous ones, on the one hand, and a richer appeal to, and reliance on, as it were, the *Elements*. The question reads:

> To discover two straight lines in the proportion of *BC* to *CD* so that the squares of the lines themselves, together with the rectangle contained under the same lines, are equal to a rectilinear figure *F* [see Fig. 9].[44]

We shall not discuss this question in any detail, simply because it contains a mistake which mars some of its inferences. We shall only call attention to the fact that it is full of ostensible algebraic operations foreign to the *Elements*, despite the fact that Bonasoni sets up his algebraic structures precisely on the *Elements*. His is true geometric algebra, not the one *invented* for reading Euclid by modern, mathematically minded historians.

What remains to be done now is to select a few representative examples from Book III of the *Algebra Geometrica*, in order to establish its overwhelming algebraic character, and thus the true character of the entire book.

The First Question of the book deals with the quadratic equation (in our notation) $x(x+B) = B*C$, the solution of which is reduced to the First Problem of Book One.

The Seventh Question, whose solution, Construction (based on the Third Problem of Book One discussed above), and Demonstration we shall not follow in detail, is interesting, however, because it points out explicitly that a quadratic equation can have, in specific problematic contexts, either one or two acceptable solutions and, most significantly, because it calls attention to the complete equivalence of geometrical and numerical problems and solutions, something inconceivable in Greek geometry:

> The same thing also happens with the composite canon [*i.e.*, this type of equation in which "squares and numbers equal things"] in numerical algebra, to wit, where a census [*i.e.*, a square] with a number is equated to the res, because when the number is subtracted from the square of half the res, and the square root of the remainder is taken, half of res is added to the root or subtracted from it. [These are

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[44] *Ibid.*, p. 41.

exactly the operations appearing in the quadratic formula.] For both addition and subtraction satisfy. And, although we are very often allowed to use both addition as well as subtraction in questions, nevertheless sometimes only addition is suitable, and other times only subtraction.[45]

Furthermore, in the continuation of the question a number of specific numerical equations of the same type (census and numbers equal res) are solved, by a succession of operations matching those of the quadratic formula. Thus, the problem of finding a number which when multiplied by 8, while 12 is subtracted from the product, makes the remainder equal to the number's own square leads to the equation $8x - 12 = x^2$, which is solved in the following successive steps:

$8x = x^2 + 12$; $(8/2)^2 = 4^2 = 16$; $16 - 12 = 4$; $\sqrt{4} = 2$; $4 + 2 = 6$; $4 - 2 = 2$. In this case, there is a twofold answer to the question.

If one has to find a number that, when subtracted from 10, "makes a remainder whose square is equal to the product made by the multiplication of the number itself into 9",[46] *i.e.*, solve the equation $(10 - x)^2 = 9x$, the procedure corresponds entirely to the following steps:

$100 - 20x + x^2 = 9x$; $100 - 20x + x^2 - 9x = 0$; $x^2 - 29x + 100 = 0$; $x = 14.5 \pm \sqrt{(14.5^2 - 100)}$; $x = 14.5 \pm \sqrt{(210.25 - 100)}$; $x = 14.5 \pm 10.5$; $x_1 = 25$; $x_2 = 4$. In this case, only the smaller root is appropriate, since one cannot subtract 25 from 10, *i.e.*, the + sign before the radical is inappropriate.

The last example is one in which the + before the radical is the only adequate sign for the specific problem; one must find a number such that when 6 is subtracted from it, the square of the remainder is equal to the product of 8 and the number. The corresponding equation is $(x - 6)^2 = 8x$ and the two roots are 18 and 2. Clearly, one cannot subtract 6 from 2, and so only 18 is adequate, corresponding to the sign + before the radical.[47]

It should be obvious to the unbiased reader that Bonasoni's book is indeed worthy of its title, which modern usurpers have snatched away unceremoniously and bestowed anachronistically on classical Greek mathematics. The conceptual chasm separating the *Elements*

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[45] *Op. cit.*, pp. 70–71.
[46] *Ibid.*, p. 73.
[47] *Ibid.*, pp. 74–75.

from the *Algebra Geometrica* can only be bridged retrospectively, *i.e.*, ahistorically. Such a bridge will lead the historically minded wayfarer to sure drowning in the troubled waters it spans.

2. *Strength and Weaknesses of Gemetrical Algebra*

What are the strengths, if any, of the methodology encapsulated in the label "geometrical algebra"? Its chief strength and, by the same token, its main weakness consists in its mathematical adequacy. It is a fact that the algebraic translation of geometrical texts undertaken by the geometrical algebraists is accurate mathematically. So what else is needed? Isn't mathematical accuracy the necessary and sufficient condition for the translation of a mathematical text? What else beyond mathematics does a mathematical text contain? To answer these questions, it is necessary to distinguish analytically between two intertwined and practically inseparable aspects of any text, including mathematical ones, namely the form and the content of the text in question.

The substance of the text, its content, is always embedded in a certain form, which is the idiosyncratic manner in which the content presents itself. It is a truism that there is no content without form or form without content, and that these two indissoluble aspects mutually condition one another and neither is immune to change.

It follows, then, that every mathematical text is a structural unity, which should not be broken into violently by the careless translator (interpreter) who may find its peculiar form awkward and antiquated. If form and content are an inviolable unity, "preserving" the content while violating the form does violence to the text and prevents its understanding in its own right, which is, after all, the least that is required of the historian. You cannot disembowel an ancient text, by extracting, as it were, its rational kernel, its substance, its "true content", and claim convincingly at the same time that your bloody operation has succeeded in revealing "the true meaning" of the text. This so-called "true meaning" is not naked, since ideas are not born formless and floating in the buff in an ideal Empyrean, but clothed in new and typically ill-fitting and indecent garments. Undressing ancient mathematical texts of their ancient (geometrical) cloths, to get, as it were, to the nude underlying mathematical truth ends up necessarily by wrapping the nudity of mathematical truths in the entirely inappropriate algebraic garb of non-Greek mathematics. The irony of this entire fateful procedure is that while it disrobes ancient

texts of their natural clothing in order to see what's underneath, it actually ends up seeing them, necessarily, through the veil of their newly acquired garments, newly acquired that is, in the very process of disrobement (interpretation). In intellectual history, total striptease is impossible. Only God can look upon naked ideas.

That the interpretive endeavor of the geometrical algebraists has received its faithful description above will become clear from the following example.[48] Proposition II.11 of the *Elements* reads:

> To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.[49]

Euclid's purely geometrical, constructive proof amounts to the following: Let *AB* be the given line (Fig. 10). Construct the square on *AB* (by I.46). Bisect *AC* at *E* and join *BE*. Extend *CA* to *F*, so that *EF* = *BE*. Construct the square *FH* on *AF*. The claim is that *H* divides *AB* according to the demand of the enunciation, *i.e.*, the rectangle whose sides are *AB*, *BH* equals in area the square built on *AH*.

And now the proof. Line *AC* was bisected at *E* and extended to *F* (*i.e.*, *AF* was added to it), exactly the setup of II.6. Consequently, the rectangle contained by *CF,FA* together with the square on *AE* is equal to to the square on *EF*. But *EF* = *BE*. Hence, rect. (*CF,FA*) along with sq. (*AE*) = sq. (*BE*). But, by I.47, sq. (*AB*) along with sq. (*AE*) = sq. (*BE*). Hence, rect. (*CF,FA*) along with sq. (*AE*) = sq. (*AB*) along with sq. (*AE*). Therefore, rect. (*CF,FA*) = sq. (*AB*). But rect. (*CF,FA*) = rectangle *FK* and sq. (*AB*) = square *AD*. Hence, *FK* = *AD*. Subtracting from both the rectangle and the square what they have in common, *i.e.*, rectangle *AK*, one is left with *FH* = *HD*, which amounts to rect. (*AB,BH*) = sq. (*AH*), q.e.f.

There can be no doubt about the plainly geometric character of this constructive proposition. Its protagonists are points, lines, rectangles, and squares. Its "operations" involve, beyond the bisection of a line and the extension of given lines, constructing squares *on* known lines and considering rectangles *contained* by given sides.[50]

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[48] Numerous other examples can be found in Unguru's works cited in nn. 7, 8 *supra*.

[49] T. L. Heath (n. 10, *supra*), vol. 1, p. 402.

[50] There is a world of difference between building the square *on* a given line, dealt with in prop. I.46, and taking the square *of* a given magnitude, which can also be a line! The former is a geometrical construction, while the latter is an arithmetico-algebraic operation; the former is Greek, the latter barbarian. The same

These are (and we use the term advisedly) blatantly geometric procedures. There is nothing metric or algebraic about them. This is why, in a lucid and convincing article, developing ideas found in earlier articles by Unguru and by Unguru and Rowe,[51] Ivor Grattan-Guinness is entitled to say:

> In his geometry, Euclid *never* multiplies a magnitude by a magnitude; for example, the line of length b is never multiplied by itself to produce the square b^2. . . . [no]where . . . in the *Elements* . . . is it stated, assumed, or proved that the area of the square is the square of a side. . . . nowhere are area *formulae* involved. . . .
>
> In other words, in Euclid's geometry *the square* on *the side is not the square* of *the side, or the side squared*; it *is* a planar *region* which has this size. In the same way, he never construes the side of a square as the square root of the square; a square can *have* an associated side . . .
>
> [In the method of application of areas] no theorems about areas *as areas*—that is, quantities with arithmetically expressible properties—are involved.
>
> This eschewal of geometrical products seems to refute the reading as geometric algebra on its own . . .[52]

The preceding discussion, it seems to us, has actually answered the questions raised at the beginning of this chapter. The only merit of the methodology embodied in "geometrical algebra" is that it is the incarnation of a *possible* interpretation of ancient Greek mathematical texts. It is a possible, but wrong, reading of those texts. It is a translation of geometrical into algebraic language, *i.e.*, it is historically a mistranslation, which no Greek mathematician could have understood. It clarifies matters *for us*, at the expense of distorting them. It is the outgrowth of a Platonic metaphysics that sees mathematical ideas as disembodied beings, pure and untainted by any idiosyncratic features. It is, therefore, ahistorical by definition.[53] It assumes that mathematical facts are not only eternal and changeless, but also that they *actually* manifest, ready for inspection, all their potentialities; as a matter of fact, the distinction between potentiality

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fundamental distinction applies to the consideration of, and the argumentation about, the rectangle *contained by* its two sides versus the *multiplication of* the two numbers measuring the sides.

[51] Especially, "On the Need" and "Does the Quadratic Equation Have Greek Roots?"; see full reference in nn. 7, 8 *supra*.

[52] "Numbers, Magnitudes, Ratios, and Proportions in Euclid's *Elements*: How did He Handle Them?", *Historia Mathematica*, vol. 23 (1996), pp. 355–75, at 360–61. This article is one of the best appearing in the wake of the "On the Need to Rewrite" article.

[53] Unguru, "History of Ancient Mathematics", *passim*.

and actuality, a distinction without which history as a discipline cannot exist, collapses in the hands of the practitioners of geometric algebra. In their hands, the history of mathematics becomes the mathematics of history. As Grattan-Guinness has it: "The underlying philosophy behind the interpretation of the *Elements* [and of Greek mathematics generally] as geometric algebra, common to mathematicians' and sometimes even historians' understanding of the history of mathematics in general, is *historical confirmation theory*: Suppose that a theory T_1, created during epoch E_1, is followed by a later theory T_2 in epoch E_2; then at epoch E_3 the (non-)historical reading is proposed that T_1 was conceived by its creators as an *intended* draft of T_2, and therefore confirms it. Roll on history, deterministically."[54]

C. *The Influence of "Geometrical Algebra" in the Interpretation of the* Conica

It is clear from the preceding (and this is not "merely" an historical point, but also an elementarily logical one) that "geometrical algebra" could not, and did not, come into being before the birth of algebra. And, though the beginnings of arithmetico-algebraic calculations may be found in Mesopotamian clay tablets, in ancient Greece, which obviously knew practical and theoretical arithmetic (logistics and arithmetic), an algebra worthy of the name cannot be identified before Diophantus and his *Arithmetica*, *i.e.*, not before the third century A.D. Though we saw burgeonings of what retrospectively can be recognized as geometrical algebra in Heron, classical Greek mathematics is not tinged by this hybrid and forbidden procedure.

In modern times, "geometrical algebra" became the main interpretive tool of much of Hellenistic mathematics. As early as the late eighteenth and early nineteenth century, one of the first modern "professional" historians of mathematics, J. E. Montucla, approaches the *Elements* as an algebraist.[55] F. Peyrard, the known editor and translator of Euclid, Archimedes, and other Greek mathematicians, falls prey to the same tendency and is praised for it by the famous and influential genius of J. L. Lagrange. G. H. F. Nesselmann, in

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[54] *Op. cit.*, pp. 371–72.

[55] J. F. Montucla, *Histoire des mathématiques*, 4 vols. (Paris, Agasse: 1799–1802), vol. 1, p. 278; a reprint edition was published in Paris by Blanchard in 1960.

his *Die Algebra der Griechen* (Berlin, 1842),[56] sees Book II of the *Elements* as arithmetical, not algebraic, in character, which does not prevent him, however, from transcribing the first ten propositions of the book in algebraic symbolism! Nesselmann's book, with its trichotomic, and problematic, classification of the historical development of algebra,[57] can be taken as marking the time when it began to become customary to see much of Greek mathematics as algebra in disguise. The main perpetrators of this modern historical Masque are a Frenchman and a Dane, Paul Tannery and Hieronymus Georg Zeuthen.

Tannery speaks "De la solution géométrique des problèmes du second degré avant Euclide"[58] and considers Book X as a theory of irrationals that is nothing less than "the detailed account of the geometrical solution of the biquadratic equation and the beginning of the solution of the triquadratic, together with the invention of a nomenclature meant to make up for the lack of [algebraic] notations."[59] In an historical appendix written for his brother's, Jules, *Notions de Mathématiques*, he states emphatically that while the algebraic symbolism of the ancient Greeks developed laboriously, they were already in possession of a geometric symbolism as early as the fourth century B.C., a symbolism enjoying all the advantages of the use of letters in Viète's analysis, at least for the second and third powers. He goes on: "Since then, and probably since the time of the first Pythagoreans, they could constitute a true geometrical algebra for the first degrees, *in full conscience that it conformed exactly to numerical operations.*[60] Then he goes on to say: "Though the Greeks did not rise to the general concept of coordinates, their manner of considering the conics is entirely analogous to our analytical geometry. . . . *The equation which they establish amounts to the modern general form:* $y^2 = px \pm p/a*x^2$ . . ."[61] He continues: "The procedures of transformations of coordinates are imperfect with the ancients, as a result of the lack

[56] Reprinted at Frankfurt in 1969 by Minerva.

[57] Cf. Leon Rodet, *Sur les Notations Numériques et Algébriques antérieurement aux XVI^e Siècle* (Paris: Ernest Leroux, 1881), pp. 56–70, *passim.*

[58] *Mémoires Scientifiques*, vol. 1, *Sciences Exactes Dans L'Antiquité*, J. L. Heiberg & H. G. Zeuthen, eds. (Toulouse: Edouard Privat and Paris: Gauthier-Villars, 1912), pp. 254–280.

[59] *Ibid.*, p. 263.

[60] *Ibid.*, vol. 3, p. 167, italics added.

[61] *Ibid.*, p. 168, italics added.

of a general conception of the problem . . . Nevertheless, these procedures do exist."[62] Such examples could be multiplied *ad nauseam.*

This brings us to Zeuthen, who is the main hero of our story. H. G. Zeuthen (1839–1920), a learned mathematician and a student of Michel Chasles, the master of enumerative methods in geometry, was cognizant of classical languages and, as a natural extension of his *mathematical* interests, delved prolifically, imaginatively, and proleptically into the history of ancient (primarily Greek), medieval, and early modern mathematics.[63] His views on geometrical algebra are spread throughout his various works, but their most cogent and complete expression appears in the first two works quoted in the last note. Thus, in *Die Lehre*, Zeuthen entitles his first chapter "Voraussetzungen und Hülfsmittel; Proportionen und geometrische Algebra". It is there that, in a paragraph remarkable for its *non sequiturs*, Zeuthen says that though the Greeks did not possess the concept of a system of coordinates, they nevertheless used "rectangular and oblique coordinates", and though Algebra was unknown to them, the historian must establish what they used in its stead.[64] He continues by saying that the Greek theory of proportions ". . . contains theorems which allow the carrying out of the most important algebraic operations . . ." and that "In this manner they and we dispose of a machinery with the help of which one can express the constitution of algebraic magnitudes."[65] Furthermore, after the discovery of incommensurability by Pythagoras or one of his disciples, ". . . the direct use of numbers and of their attached proportions came to a halt in geometry, which had rightful claims to rigor . . . Still, it could not have been the case that numbers and numerical proportions were not used in geometry, even if with the awareness of the need to prove the gained results in a different way afterwards [!], in order to make them acceptable."[66]

But the modern manipulations of proportions are a direct outgrowth of the existence of a symbolic mathematical language that

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[62] *Ibid.*

[63] His most important and representative historical works are: *Die Lehre von den Kegelschnitten im Altertum* (Copenhagen, 1886); *Geschichte der Mathematik im Altertum und Mittelalter* (Copenhagen, 1896); *Geschichte der Mathematik im XVI. und XVII. Jahrhundert* (Leipzig, 1903). Our book is, in a real sense, a response to *Die Lehre.*

[64] *Die Lehre*, p. 2.

[65] *Ibid.*, p. 4.

[66] *Ibid.*, p. 5.

makes the operations involved easily graspable and retainable in memory. On the other hand, Antiquity, which had no symbolic language, "... disposed nevertheless of an aid *for the illustration of these and of other operations, namely, the geometrical representation and treatment of general magnitudes and of the operations to be undertaken with them.*"[67] And now comes the pregnant statement:

> In this manner, a geometrical algebra, as one could name it, evolved, since it could handle general magnitudes, irrational as well as rational, and since it used different means than ordinary language to make clear its procedures and impress them on memory. This geometrical algebra had reached in Euclid's times a stage of development which enabled it to master the same tasks *as our algebra, as long as they did not transcend the treatment of expressions of the second degree,* a domain suitable also for its application to the study of conic sections. Such an application corresponds to the application of our algebra in analytical geometry.[68]

Having dealt with the ancient theory of proportions, Zeuthen passes on to "geometric algebra" proper and establishes to his own satisfaction that the Greeks had the means to represent the equation $\alpha x + \beta y + \gamma z + \ldots = d$ as follows:

> If α, β, γ... are represented as ratios between given straight lines, while $x, y, z \ldots d$ are themselves represented as straight lines, then the equation [above] would thereby be representable, if consecutive segments were taken on a straight line which would have respectively the ratios α, β, γ... to x, y, z... The distance between the point of origin and the point reached after successive allotments of the respective segments would then be d. One could proceed in a similar manner when there are other signs in the equation. Precisely as we, in the presently customary representation, must remember the meaning of each letter, the ancients had to remember which segments were assigned; *but then the ancients had at their disposal, exactly as we do, a representation of the equation....* By means of such a representation, are solved equations of the first degree in a manner which has much in common with our own algebraic handling.[69]

It is significant to observe that, even while assuming the substantive correctness of the above description, what Zeuthen has done therein is to show that one could start with the equation and *thereafter* translate it into the language of Greek geometry. This, of course, proves

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[67] *Ibid.*, p. 6.
[68] *Ibid.*, p. 7.
[69] *Ibid.*, pp. 9–10, italics added.

nothing pertinent to the ancient Greeks, but the procedure is important, since it puts into relief everything that is wrong with "geometrical algebra". The hermeneutical sins of commission and omission are simply more blatant in this procedure than in the traditional one, which starts with the Greek geometrical text and translates it into a language unavailable to the ancients and which, moreover, trivializes and cheapens their tremendous achievement; both translation procedures, whatever their directions, are apostasies, since under the guise of clarification they distort and utterly misrepresent.

The first chapter of Zeuthen's *Die Lehre* contains in its 38 pages a complete and concise rendering of all the bewitchingly insightful, misleading, and influential tricks associated with "geometrical algebra". We cannot survey them all here and this is also not necessary as most of them were already dealt with in the "On the Need" article, in the *Libertas Mathematica* studies, and in other items mentioned before.[70] What needs to be done here is to report briefly on the "insights" of "geometrical algebra" as they apply to the theory of conic sections.[71]

Zeuthen, and all those who have embraced his views, including Heath, Neugebauer, van der Waerden, etc., etc., think that Book II of the *Elements* and parts of Book VI, namely propositions VI.27–29, contain the geometrical solution of the quadratic equation. Zeuthen also thinks that Apollonius uses in the *Conics* the same means, though in slightly modified form,[72] to solve quadratics, which are here intimately associated with the conic sections.

Having shown to his own satisfaction how Book II treats quadratics of special types,[73] Zeuthen shows, by means of unacceptable

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[70] One other item worth mentioning, as it relates directly to the *Conics*, is S. Unguru and M. Fried, "On the Synthetic Geometric Character of Apollonius's *Conica*", *Mathesis*, vol. 12 (1996), pp. 148–223.

[71] "Briefly", because our entire book is a stock-taking of, and critical response to, Zeuthen's *Die Lehre.*

[72] *Ibid.*, p. 31.

[73] It is Zeuthen's conviction that the Greeks were familiar with quadratic equations already before Euclid, that this familiarity included the knowledge of how these equations should be used in both geometry and arithmetic, and that Euclid merely gave expression to this old knowledge in Book II of his *Elements.* He goes on to say: "Euklid und seine Vorgänger in der Abfassung von Elementen haben also die Sätze dieses Buches aufgestellt und bewiesen in dem vollen Bewustsein, dass sie auf dieselbe Weise benutzt werden konnten, wie es über ein Jahrtausend später von arabischen Schriftstellern, die sich eben auf Euklid stützen, geschah, entweder im Anschluss an uns unbekannte Überlieferungen, oder mit einem richti-

algebraic manipulations, discussed in detail in the *Libertas Mathematica* studies, how propositions VI.27–29 solve allegedly the general quadratic. We shall not repeat here the analysis appearing in the two lengthy texts comprising the "Does the Quadratic Equation Have Greek Roots?"-articles. Suffice it to say that Zeuthen succeeds in showing that, according to the canons of geometrical algebra, Euclid solves in proposition VI.28 the equation $ax - c/d{*}x^2 = B$, for which he finds the root, $c/d{*}x = a/2 - \Re[(a/2)^2 - c/d{*}B]$, where $\Re$ represents the symbol for the square root. By the same ahistorical canons, VI.29 solves the equation $ax + c/d{*}x^2 = B$, which is nothing but " the immediately straight translation" of the Euclidean proposition,[74] and finds the root, $c/d{*}x = \Re[(a/2)^2 + c/d{*}B] - a/2$. In this procedure, claims Zeuthen, we find "a rigorous solution of the quadratic equation with three [different] coefficients."[75] And since it is clear that the numerical solution of numerical equations with the square member x^2 was already a pre-Euclidean achievement, one must assume that it was also a pre-Euclidean accomplishment to solve equations of the type $ax^2 \pm ax \pm B = 0$, by reducing them to the form where the coefficient of x^2 was 1, for example, by the substitution $ax = z$![76]

Is there, then, anything about quadratic equations which the ancient Greeks in their algebraic prowess could not solve? Yes, there is, but even this limitation is *not* absolute: "Still, these equations[77] could not be represented in the two-dimensional geometrical algebra in a generally valid manner, *i.e.*, in a manner valid also for irrational values of x, because of the three factors in ax^2, without the benefit of the theory of proportions of Eudoxus."[78]

All this, of course, and much more, Apollonius knew. He was, after all, a master algebraist. At least this is how he is seen by the Danish mathematician and master of enumerative geometry after

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gen Blick für das, wozu Euklids Sätze überhaupt nützlich sind" (*op. cit.*, p. 29). Incredible as it is in its blatant anachronism, offensive ahistorism, and in its classical display of what Grattan-Guinness called "historical confirmation theory" (cf. *supra*, nn. 31, 33), this position is only drawing the logical conclusions following from the methodology of "geometrical algebra".

[74] *Ibid.*, p. 30.

[75] *Ibid.*, p. 31.

[76] *Ibid.*

[77] *I.e.*, equations in which the coefficients of x^2 and x are identical.

[78] *Ibid.* Here too an attentive reading of the quoted passage shows that Zeuthen implies that the Greek mathematician started somehow (?) with the disembodied equation which he later "represented" geometrically. Now, this is geometrical algebra!

Chasles, H. G. Zeuthen.[79] By using these traditional methods, Apollonius came up, according to Zeuthen, with the following equations for the hyperbola and ellipse respectively, where x and y are the coordinates of a movable point on the curve represented in a system of coordinates in which the abscissas are measured on the diameter, and the ordinates on the tangent at its extremity:

$$y^2 = px \pm p/a*x^2$$

This reading of the *Conica* is gained through the discrimination and flair acquired from the insights of geometrical algebra coupled with an idiosyncratic view of the traditional Apollonian diagrams, in which the cone and its sectioning planes are dropped altogether, as are all those diagrammatic elements which only "complicate needlessly" the given situation, from the standpoint of the geometrical algebraist. It is enough to compare Book One of the *Conica* with Zeuthen's treatment of it in *Die Lehre*,[80] to realize the dramatic discrepancies between the geometric approach and the geometric-algebraic therapy the former undergoes in Zeuthen's dexterous hands. The cone, the originator of the conics, practically disappears, hence the Apollonian diagrams are streamlined drastically, to the point of no recognition, and one is left with the bare quasi-algebraic relationships between magnitudes whose conic origins are merely accidental. Zeuthen will not hesitate to group propositions under the same heading and to deduce them from the same simplified diagram, against Apollonius's way, since what matters in his approach are disembodied quantitative relationships between numerical magnitudes, which happen to appear in linear garb. In Zeuthen's hands, Apollonius becomes more abstract, generalized, systematic, more concerned with structural considerations than with the way of proving specific properties of the individual conics tied always to the generating cone.[81] In other words, Apollonius becomes a crypto-algebraist.[82]

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[79] Conceivably, Zeuthen's main contribution to mathematics is his *Lehrbuch der abzählenden Methoden der Geometrie* (Leipzig and Berlin, 1914).

[80] "Dritter Abschnitt", pp. 63–85 and also parts of "Vierter Abschnitt", pp. 86–104.

[81] There are, of course, propositions in the *Conica* in which the cone as such does not appear; but it is always understood that the cone is necessarily in the background and the last propositions of book one, 52–60, show how to construct all the sections from given initial conditions, by fitting them into their generating cones. In other words, when the sections occur by themselves, we know how to supply the generating cone.

[82] The centrality of the cone in Apollonius's universe of discourse will become

The "immediately direct" connection between the equations and the diagrams à la Zeuthen is as follows: $x = AC$, $y = CD$. The square on CD is equal to the rectangle AF which is applied to $AE = p$, in such a manner that that the defect (in the case of the ellipse) or the excess (in the case of the hyperbola) are similar to the rectangle whose sides are $AE = p$ and $AB = a$. This is achieved by having the diagonal EF pass through B. Consequently, the rectangle $EF = p/a{*}x^2$, while rectangle $CE = px$. Hence, hocus-pocus preparatus, $y^2 = px \pm p/a{*}x^2$, *q.e.d.*[83]

Zeuthen goes on to claim that the Apollonian diagrams, accompanied by words as they always are, have the same heuristical value for theoretical investigations as the algebraic representation![84] This is why it is so easy to see how highly appropriate geometrical algebra is for the investigation of conic sections and how effortlessly and naturally it lends itself to the nature of these curves.[85] Indeed, a great many properties of the conics reveal themselves readily in yielding to the efficient scalpel of an analytical-geometrical scrutiny, when one relates the conic section to various systems of coordinates.[86] Thus, to give just one example, the most important properties of conjugate diameters come to the fore as a result of the fact that there are infinitely many (!) coordinate systems for which the equation of a conic takes the form $y^2 = px \pm p/a{*}x^2$. Moreover, if, in considering the location of the section on the cone, one selects for it an adequate equation, say, the vertex equation relative to an axis, then the change to new systems of coordinates takes place through linear substitutions in the previously found quadratic equation. The algebraic operations of second degree used in this procedure are precisely those with the geometrical form of which the Greeks were highly familiar, as we know from the second book of the *Elements*.[87] The historical farce goes on: "The substitution coefficients and the coefficients of the terms of second degree in the equations are not expressed by means of lines, which would result in making the equations, in which these magnitudes are expressed as lines, of a degree higher than two,

abundantly clear in the following chapter; for Zeuthen's diagrams, the reader should view Fig. 11.

[83] *Ibid.*, pp. 32–33.

[84] *Ibid.*, p. 33.

[85] *Ibid.*, p. 34.

[86] *Ibid.*

[87] *Ibid.*

but rather by means of ratios, and these ratios are introduced as a rule in as clearly identifiable forms as [the equations $y^2 = px \pm p/a*x^2$]. At the same time, everything is so calculated as to unify as many terms as possible, so that the equations of the curves express quite frequently that . . . two variable surfaces are of equal size, or, also, that one variable surface preserves a constant value."[88] Zeuthen continues in the same vein: "In the resolution of the equations of second degree, one also disposed of a tool for determining the points of intersection [of the curves] with straight lines, and, in the diorismus pertaining to quadratic equations (Euclid, VI.27), the condition of tangency, and therewith a means for determining tangents."[89]

Continuing his discussion of the applicability of geometrical algebra to the theory of conic sections, Zeuthen points out that the equations of the conics must always be seen as linear equations between areas. However, if one were to submit to an ancient Greek mathematician, instead of these, the corresponding products of lines, he would have surely seen this as a sign that the proposal was put forward by somebody who did not know that not all magnitudes of the same kind have a common measure. Still, when these magnitudes are commensurable, *i.e.*, when the multiplied magnitudes can be expressed in terms of rational numbers, the Greek mathematician himself would have seen this exchange between surfaces and line products as entirely legitimate and as something that he himself was familiar with and actually practiced! Furthermore, what follows from this is that, most likely, he practiced the same interchange also in cases where rationality was absent, in order to obtain approximate results, though this transcended the bounds of rigorous geometry!

Zeuthen sums up his discussion by observing that it is geometrical algebra that explains why only the theory of conic sections came to full fruition in Antiquity, while the investigation of higher curves remained sporadic at best. This was so because, in the latter case, it did not suffice to express the constants by means of ratios in order to bring the equation to its clearest, simplest, and most convenient form. It is true that a cubic could still be represented by means of a relation between volumes, but it is much more difficult to operate with these than it is with surfaces. "If the curve were of a still

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[88] *Ibid.*, pp. 34–35.
[89] *Ibid.*, p. 35.

higher degree, then the only means of representation of the products of more than three variable magnitudes appearing in its equation [!] were those of compound ratios, which, in this case, remained quite unwieldy."[90]

Finally, Zeuthen ends his illuminating excursus on Conics and Equations with a disarming confession: "For the sake of [accurately assessing] the role played by geometrical algebra in the ancient theory of conic sections, it was important for me, in considering these present investigations, to earn such a personal proficiency in dealing with the geometrical form of elementary algebraic operations that thinking of modern means of representation would not influence my judgment about which transformations or modes of representation may be more or less natural and suggestive."[91] Now, we have already witnessed what this assiduous training in the ancient ways of dealing geometrically with algebraic matters has done more or less to Zeuthen's historiography. However, as if to dispel any, possibly still remaining, lingering doubts about his position, Zeuthen tells us that Apollonius's text itself is one of the best training grounds for attaining a high degree of competence in geometrical algebra![92] It is the case, he goes on, that Apollonius leaves out of his proofs quite often little steps, which are entirely equivalent to the small intermediate calculations left out by the analytical geometrician. Furthermore, if we translate the Apollonian presentation into our algebraic language, then we witness that, in many instances, precisely such an intermediate calculation is left out, and that these jumps are, exactly because they shorten the otherwise broadly generous presentation, an alleviation for the modern reader. "Hence, since the Ancients possessed in the operations of geometrical algebra a well known aid which was entirely equivalent to our algebra in the domain dealing with the theory of conic sections, it is fair to assume that *it was Apollonius's opinion that his readers would, with the help of geometrical algebra, be able to verify easily those [unproved] assertions which we verify nowadays by means of elementary algebra.*"[93]

That Apollonius was a skilled geometrical algebraist is clearly the

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[90] *Ibid.*, p. 36.

[91] *Ibid.*

[92] There is here, we are afraid, more than a potential danger of a *circulus in probando.*

[93] *Ibid.*, p. 37, italics supplied.

considerate opinion of Zeuthen. It is an opinion based exclusively on a Platonic philosophy of mathematics, according to which one and the same mathematical idea remains the same irrespective of its specific manifestations; *i.e.*, to put it bluntly but accurately, Apollonius was a geometrical algebraist because Zeuthen was one when reading the *Conica*: If Zeuthen can do it, then Apollonius certainly could. Math is math is math.

Those missing steps in Apollonius's proofs, which are identified by Zeuthen as algebraic, can actually be supplied by means of other *geometrical* propositions contained in the Greek mathematical corpus, an undertaking performed indeed by Pappus in his *Collectio*. But this is unacceptable to Zeuthen, who sees it as a sign of pedantry characteristic of Pappus's times.[94] *Operations* are always better than *propositions*.[95] When Eutocius (sixth century A.D.) proves, in his commentary on the first four books of the *Conica*, auxiliary propositions necessary for a full understanding of the text, and does this "operationally", he is to be preferred to Pappus (third century A.D.), who does the same thing "propositionally", since this is how Apollonius surely must have thought.[96] "In any case, the geometrical-algebraic operations known to the ancients were actually [then] the easiest way to verify those [unproved] assertions, and this is why one is [now] justified in using the very same ancients to practice [and gain dexterity] in the same operations."[97]

Zeuthen's approach to the *Conica* is not graspable by anybody whose mathematical world is circumscribed by the ancient Greek universe of discourse. This is enough to disqualify it as a viable interpretation of Apollonius.[98] Mathematics does have a real history which differs from that imagined by Zeuthen and mathematical historians of the same ilk. History of mathematics is not the peeling off of layers of irrelevant formal lineaments in order to get to an unchangeable kernel of mathematical truth. In the real history of mathematics,

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[94] *Ibid.*, p. 38.
[95] *Ibid.*, pp. 37–38.
[96] *Ibid.*, p. 38.
[97] *Ibid.* Again, there is here an attitude of self-fulfilling prophecy that smacks of a *circulus vitiosus*.
[98] We shall deal with specifics concerning *Die Lehre* and our disagreements with them throughout this book. This chapter is meant, among other things, merely to point out objections of principle to the methodological approach embodied in "geometrical algebra".

layers of content are intimately and inseparably wedded to layers of form. A given form can accommodate only a certain content which is natural to it. Breaking the form which hides, as it were, the true content is doing violence to both form and content, which constitute an intimate and unbreakable unity. Form and content are partners unto death; for better, for worse, for richer, for poorer, in sickness, in health, etc. The first commandment of the God of History, which specifically relates to the partnership of Form and Content, decrees: "Those whom God has joined together, let no man put asunder."

Though every interpretation is a translation and though every translation is, *in extremis*, a betrayal, the legislator distinguishes always between petty and high treason. If to interpret is to translate and to translate is to betray, then every interpreter of culture is a traitor and the assignment of guilt is merely a question of the degree of treason perpetrated. The best translators are those who perform the lesser crime. What marks indelibly the best translation (interpretation) is that it does the least violence to a text, or to any other cultural product, in the treacherous process of (cultural) seduction. Notwithstanding the problematic character of translation, there is no understanding without translation; and translations can be good or bad. They can, in principle, vary from "mere" rewording, to translation proper, to transmutation.[99] Clearly, any interpretation (translation) worth its salt adds necessarily something to its object. Therefore, in assessing its qualities, one must assess the nature of what is added in the process of translation. If what is added is foreign to the original source, then, inevitably, the translation is a bad translation. Adding algebra to Greek geometry is precisely a case in point.

Algebra and geometry are different languages.[100] Greek is still another. Any specific language, say Greek, is an idiosyncratic instrument with its own robustness and vulnerability. Natural languages are not nonaligned, nonpartisan, neutral, purely abstract, supremely universal gowns that can drape and accommodate equally well any conceptual content. "To comprehend *Greek* mathematics, the historian must insinuate himself into the otherness that is the Greek mathematical

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[99] This is the trichotomous theory of signs and meaning of C. S. Peirce and Roman Jakobson; cf. George Steiner, *After Babel: Aspects of Language and Translation*, 3rd ed. (New York: Oxford Univ. Press, 1998), pp. 260–61.

[100] "On the Need", pp. 75–77, *passim*.

universe of discourse. Any procedure that aims at understanding Greek mathematics *qua* Greek phenomenon cannot adopt a hermeneutics that bodies forth the pliant, flexible, supremely malleable and moldable, indefinite, *apeiron*-like, neutral slime of the symbolic, algebraic world order. Thoughts, including mathematical thoughts, are not born naked to be later clothed in any indefinite, impartial, neutral garb."[101]

Zeuthen, and the geometrical algebraists, have put forward an interpretation that is far from being a sympathetic translation, but rather a "... transmutation (perhaps transubstantiation), *i.e.*, transposition of the original, verbal, rhetorical text into a non-verbal sign system, in this case the symbolic, algebraic system. Indeed, the vulgate of the typical modern manufacturer of historical studies in the domain of the history of mathematics has been algebra. It has been a most unhappy choice."[102]

Intra-language translation and inter-language transmutation are poles apart. The former is the methodological tool of the historian while the latter is ahistorical by definition. Moreover, algebraic symbols are monosemic, while ordinary language constructs are typically polysemic. The letters appearing in geometrical proofs are abbreviated proper names not symbols. As such they are "... both much richer and more circumscribed than the dry, exact, abstract algebraic symbols used to transcribe them and the proofs in which they are embodied. The Greek language is richer and, therefore, more ambiguous, mysterious, equivocal, and much less precise than the symbolism of algebra used to 'translate' it."[103]

The diachronic approach of the historian should replace the synchronic approach of the mathematical historian, if one wishes to write the *history* of mathematics and not the *mathematics* of history. "If by *mathesis* we mean the analytical reasonings represented by the new way of doing mathematics begun in the sixteenth and seventeenth centuries by (primarily) Viète, Descartes, and Fermat, ... then ... *Lexis non est mathesis*."[104] Until the 1970s, most historians of mathematics have been Platonists in their approach.[105] This means, among

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101 "Does the Quadratic Equation have Greek Roots?", part II, p. 45.
102 *Ibid.*, pp. 45–46.
103 *Ibid.*, p. 47.
104 *Ibid.*
105 Unguru, "History of Ancient Mathematics," p. 555.

other things, that what interested them in the mathematical past has been the eternally true, the constant. Furthermore, culture was defined 'topologically' as "a sequence of translations and transformations of constants."[106]

> This definition is . . . an *ex post facto* construct of the interpreter of culture. Speaking of constants is possible only *after* transformations were performed which, to the eye of hermeneutics, have left invariants. It makes no sense to speak of constants in the absence of change. And the historian is primarily interested in the event of change. To speak of constants, then, makes sense only because there are variables. And historical assessment involves taking stock primarily of variability. If nothing changes there is no history.
>
> Specifically, *after* algebra it is possible to find its rational kernel in geometry, thereby identifying . . . a constant; doing so, however, amounts to overlooking the very change that took place in the creation of algebra. This approach, therefore, that might sit well with the structural anthropologist is self-defeating (because blinding) for the historian.[107]

We obviously feel that H. G. Zeuthen's historiography of the conic sections in Antiquity has indeed overlooked the change that took place in the seventeenth century with the invention of analytical geometry; and this is what makes it wanting as a historical explanation. Incontrovertibly, that invention has proved to be an invention of great power, and, for that very reason, it has also proved to be a siren song for the unaware voyager on the remote and stirring waters of ancient mathematics. Zeuthen was seduced by the charming powers of algebra and analytic geometry and this fateful seduction vitiated his historiography. We must now try and move forward with a steady eye on the Apollonian text, eschewing always geometrical algebra, or any other foreign notion likely to obscure the attempt to understand Apollonius on his own terms. We shall begin with the foundations of conic sections as Apollonius develops them in Books I–III of the *Conica*.

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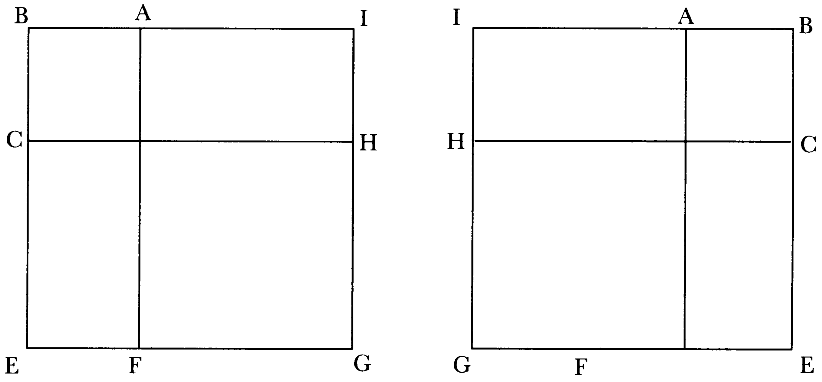
[106] Steiner, *op. cit.*, p. 426.

[107] "Does the Quadratic Equation have Greek Roots?", pp. 47–48.

Figures



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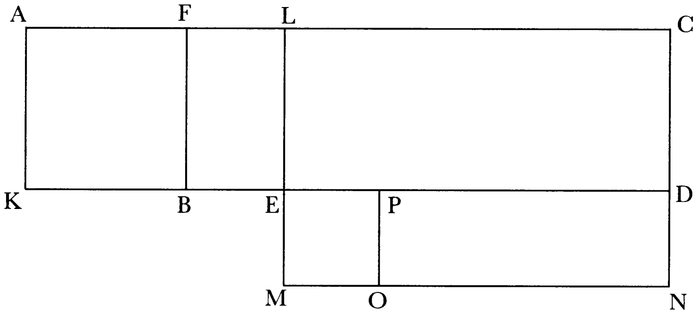

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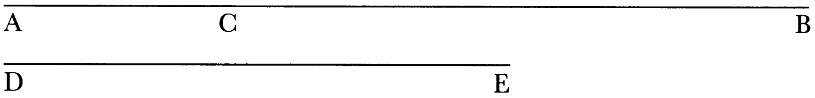

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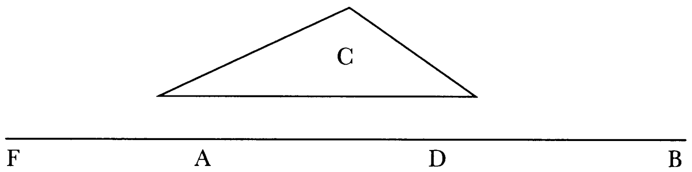

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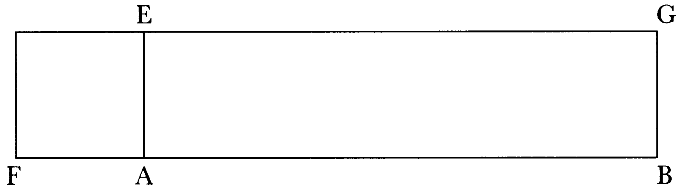

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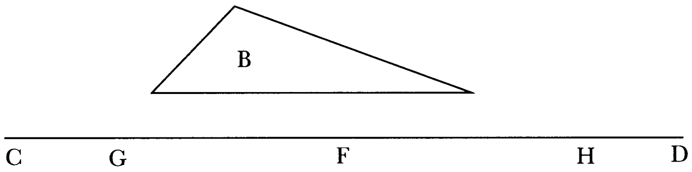

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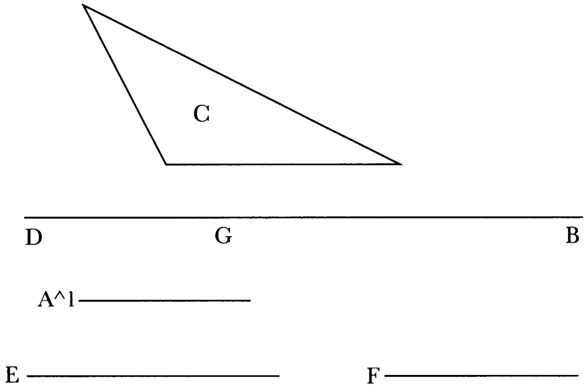

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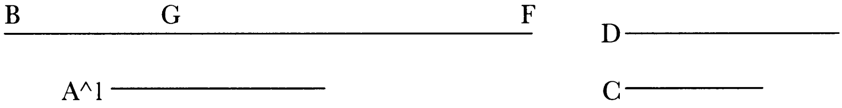

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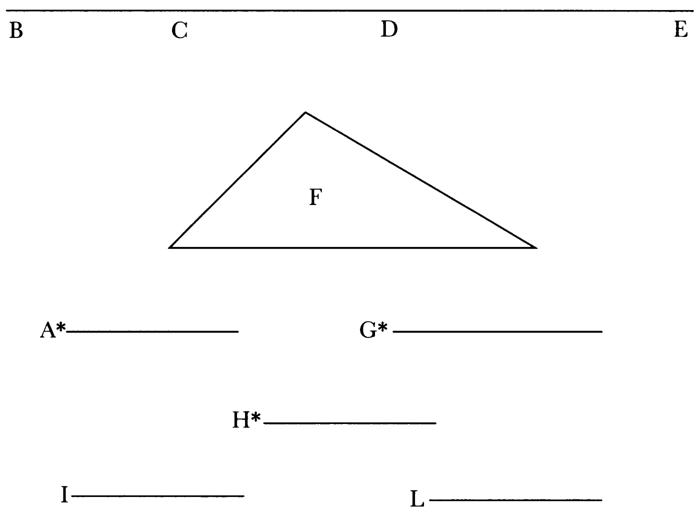

Fig. 9

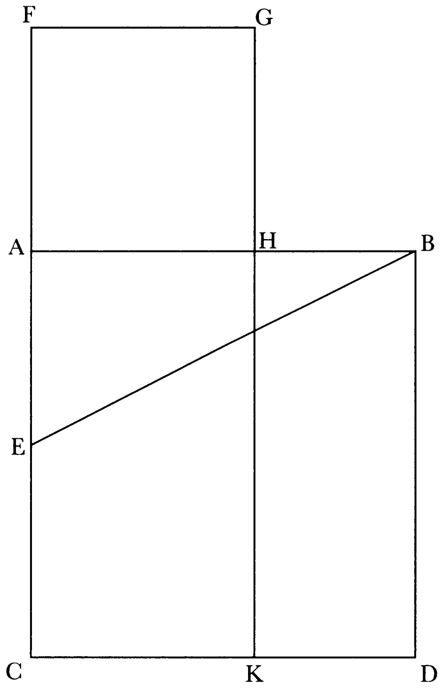

Fig. 10

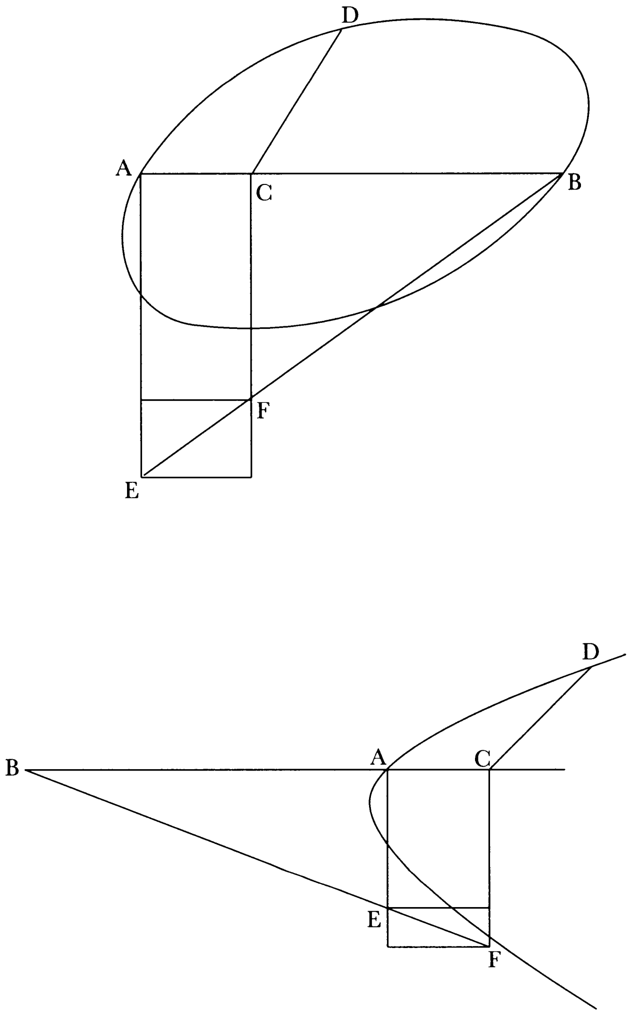

Fig. 11

CHAPTER TWO

THE ELEMENTS OF CONIC SECTIONS IN THE FRAMEWORK OF BOOKS I–III[1]

Introduction

In Apollonius's prefatory letter to Book I of the *Conica*, one reads that "Of the eight books [on the conic sections] the first four belong to a course in the elements (πέπτωκεν εἰς ἀγωγὴν στοιχειώδη)."[2] Such a "course" must, naturally, contain Apollonius's *basic* understanding of the conic sections. Therefore, to grasp the character of Apollonius's thought about conic sections one ought to begin by considering the character of his "elements." This, then, is the chief goal of the present chapter. In particular, we shall try to show that these "elements" show a geometric bias both in their inception and in their development. In the first part of the chapter, we shall look exclusively, perhaps even narrowly, at the *Conica* itself, concentrating especially on Books I–III (Book IV we shall consider separately in the next chapter). This part will be divided into three sections. Of these, the first will comprise some preliminary remarks based on the prefatory letter to Book I. In the *Conica*, as is typical of extant Hellenistic mathematical works, commentary within the text is sparse, if not nonexistent. Therefore, in this letter we have a rare opportunity to hear Apollonius himself speaking about his goals and motivations. But, alas, the letter being so short we shall have to be satisfied only with guiding questions rather than firm answers. In the second section, we shall discuss the definitions given at the start of the first Book. We shall see that these definitions give the book a setting marked unmistakably by a geometric mode of thinking. Following this, we shall consider the definitions of the conic sections themselves. The first question to answer here will be why these definitions appear only *after* the

[1] This chapter is based on our "On the Synthetic Geometric Character of Apollonius's *Conica*."

[2] Unless noted otherwise, all quotations from Books I–III of the *Conica* are from the translation by R. Catesby Taliaferro, *On Conic Sections, Books I–III by Apollonius of Perga.*

initial ten propositions rather than at the start of the book. This discussion will, we hope, not only shed some initial light on the definitions of the conic sections, but also bring out the importance, little appreciated in literature on the *Conica*, of these first ten propositions. To complete this discussion, we shall consider the peculiarity that the definitions of the conic sections appear *within* propositions. Since these propositions prove the "symptomata" of the curves, it will be necessary in this section to bring out the distinction between the definition of, say, the parabola and its "symptoma." The interpretation of the "symptomata" is crucial in the *algebraic* interpretation of the *Conica*, as discussed in our chapter on geometrical algebra. Therefore, in the concluding part of the chapter, we take a second look at this interpretation in light of our discussion here of Books I to III. Among other things, we shall examine in this part the meaning of the diameter and the ordinates. Specifically, we shall show that they are, and in what sense they are, *geometrical* elements in the context of the *Conica*, rather than things akin to the coordinate axes essential in the algebraic treatment of conics in analytic geometry.

A. *Apollonius's Introduction and the General Nature of "Elements"*

"Apollonius to Eudemus, greetings." Who is this Eudemus? We know that Apollonius knew him in Pergamum and that he died before he could receive the fourth book, as we learn in the letter accompanying that book. But other than this, we really know very little about him. Toomer says that he was the first teacher of Philonides, the geometer Apollonius mentions in the preface to the second book.[3] Whatever the case, we know something for sure about Eudemus: either he was himself a working mathematician, a teacher of geometry as Toomer says, or at least someone who was perfectly capable of understanding the contents of the books Apollonius had sent him. After all, in the preface to Book II, Apollonius tells Eudemus to "Go through [the book] then carefully and acquaint those with it worthy of sharing in such things (διέλθε οὖν αὐτὸ ἐπιμελῶς καὶ τοῖς ἀξίοις τῶν τοιούτων κοινωνεῖν μεταδίδου)." It is hardly likely that Apollonius would trust Eudemus to decide who was worthy and who

[3] G. J. Toomer, "Apollonius of Perga," p. 179.

was unworthy of sharing in Apollonius's work, if he did not think Eudemus was worthy enough himself. So this letter to Eudemus is a letter from one accomplished geometer to another and, thus, the book it accompanies is a text intended for an accomplished geometer.

That the intended reader of the *Conica* is one such as Eudemus is important to bear in mind when one considers the broad division Apollonius makes regarding the contents of the *Conica*, namely, that the first four books "belong to a course in the elements," while the latter four "are fuller in treatment (περιουσιαστικώτερα)."[4] One does not expect an *elementary* treatment of a subject to be of great interest to a professional in the field. Yet, in relating the contents of the *Conica* to Eudemus, Apollonius shows no sign that he expects Eudemus to be less interested in the "course in the elements" than in the other books. On the contrary, Apollonius's description of the "elementary" books, Books I–IV, is far more extensive than that of the later books, Books V–VIII; counting lines of text, the description of the former is almost *five* times as long as the latter. This can only show that Apollonius expects Eudemus to be as interested, if not more, in the "elementary treatment," as in the rest. A strange state of affairs? It may only mean that we are missing what Apollonius intends by "elements" and "elementary."

According to Heath, the elementary nature of the first four books distinguishes them from the rest by the "fact that the former contain a connected and scientific exposition of the general theory of conic sections as the indispensable basis for further extensions of the subject in certain special directions, while the fifth book is an instance

[4] There is no complete agreement on how to translate περιουσιαστικώτερα Taliaferro has it as it is written above, whereas Ivor Thomas translates it as "thrown in by way of addition" (*Greek Mathematical Works* II, 285). Heath, who translates περιουσιαστικώτερα "by way of surplusage" (*Apollonius of Perga*, p. lxxi), has this to say, "περιουσιαστικώτερα has been translated 'more advanced', but literally it implies extensions of the subject beyond the mere essentials. Hultsch translates 'ad abundantiorem scientiam pertinent', and Heiberg less precisely 'ulterius progrediuntur'" (*ibid.*). Heath's reading of περιουσιαστικώτερα as "something extra" is consistent with the basic meaning of περιουσία, which Liddell and Scott give as "*that which is over and above* necessary expenses, *surplus, abundance, plenty*." Yet, Taliaferro's reading, however, works well with Apollonius's description of Book V as dealing *fully* (ἐπὶ πλέον) with minima and maxima. In the end, the final interpretation of περιουσιαστικώτερα must come together with that to which it is contrasted, namely, "elementary (στοιχειώδης)," and, no doubt, the ambiguity of the one is linked to that of the other. The present chapter shall discuss and explore the one side of the coin, "elements" and "elementary," leaving the other side of the coin for later.

of such specialization . . ."[5] Heath also calls the first four books a "text-book or compendium of conic sections," and the last four books "a series of monographs on special portions of the subject."[6] Toomer adopts the same kind of image when he writes, "[Apollonius's] aim was not to compile an encyclopedia of all possible theorems on conic sections, but to write a systematic textbook on the 'elements' and to add some more advanced theory which he happened to have elaborated."[7] These characterizations are not entirely wrong. In his letter, Apollonius does speak about correcting and revising the work, about it being at one time in a different form. He seems to be interested in his text being readable and coherent, "a connected and scientific exposition . . ." But does this make it a "text-book"? Toomer is most likely correct in saying that the *Conica* was not to be "an encyclopedia of all possible theorems," if only because one can find examples of theorems *not* in the *Conica* that Apollonius certainly could have known, and probably did know.[8] But not being an encyclopedia does not necessarily imply that the *Conica* is a textbook, at least not in the modern sense of the word. The *Conica* is not a book for a student, for a not-quite-yet mathematician, rather it is a book, as we have seen, for one like Eudemus, a professional mathematician. In this sense, then, it is, at least, misleading to describe the *Conica* as a "textbook." However, a professional mathematician can still be a learner, of course, and Apollonius's letter gives us a hint of the sort of learning in which a Greek mathematician is engaged. For having described the contents of Book II, Apollonius says "And what I call diameters and what I call axes you will know (εἰδήσεις) from this book." We note that diameters and axes are both *defined* in the *first* book, not in the second, so that "to know" what Apollonius calls diameters and axes is to know more than their definitions. As we shall discuss later, to know what are called "diameters" and "axes" is to know about them and, in particular, to know enough about

[5] Heath, *Apollonius of Perga*, p. lxxvi.

[6] *Ibid.*, pp. lxxvi–lxxvii.

[7] Toomer, "Apollonius of Perga," p. 187.

[8] For instance, suppose ABC is a parabola, BD is a diameter, and L is its upright side (see fig. 1). Let AC be drawn ordinatewise to BD, MC tangent at C, MOX drawn parallel to BD, and O the point of intersection of MOX and the parabola. Then CA:AX::MX:XO. This is prop.5 in Archimedes's *On the Quadrature of the Parabola*, and it is used by Archimedes in the first proposition of his *Method*. However, this proposition is not in the *Conica*, nor is it an *immediate* result of another proposition in the *Conica*, although one *can* prove it by means of propositions there.

them in order to construct them, to present them to the eye; a complete picture must emerge before Eudemus can say he has learned what Apollonius calls "diameters" and "axes."

Following this same train of thought, we can ask what Apollonius intends by a "course in the elements" by asking what it means for Eudemus to *learn* the "elements." Writing on the purpose of Euclid's *Elements*, Proclus says "If we start from the elements, we shall be able to understand the other parts of this science; without the elements we cannot grasp its complexity, and the learning of the rest will be beyond us. The theorems that are simplest and most fundamental and nearest to first principles are assembled here in a suitable order, and the demonstrations of other propositions take them as the most clearly known and proceed from them."[9] Thus, the "elements" are the basic and formative parts of the science, and are, accordingly, the foundation on which the science rests. Proclus compares them to the letters of a written language, noting that the same word in Greek is used both for "letter" and "element," στοιχεῖον.[10] He distinguishes, however, between theorems which are *elemental*, that is, are *elements* (στοιχεῖα), and ones which are *elementary* (στοιχειώδη).

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[9] Glenn R. Morrow, *Proclus: A Commentary on the First Book of Euclid's Elements*, pp. 58–59.

[10] Aristotle uses a similar analogy in his definition of "elements" in Book Δ of the *Metaphysics*, the "philosophical lexicon." In general, Aristotle's discussion has, we think, echoes in Proclus, so we shall quote it in full. "'Element' means (1) the primary component immanent in a thing, and indivisible in kind into other kinds; e.g. the elements of speech are the parts of which speech consists and into which it is ultimately divided, while *they* are no longer divided into other forms of speech different in kind from them. If they *are* divided, their parts are of the same kind, as a part of water is water (while a part of the syllable is not a syllable). Similarly those who speak of the elements of bodies mean the things into which bodies are ultimately divided, while *they* are not longer divided into other things differing in kind; and whether the things of this sort are one or more, they call these elements. The so-called elements of geometrical proofs, and in general the elements of demonstrations, have a similar character; for the primary demonstrations, each of which is implied in many demonstrations, are called elements of demonstrations; and the primary syllogisms, which have three terms and proceed by means of one middle, are of this nature. (2) People also transfer the word 'element' from this meaning and apply it to that which, being one and small, is useful for many purposes; for which reason what is small and simple and indivisible is called an element. Hence come the facts that the most universal things are elements (because each of them being one and simple is present in a plurality of things, either in all or in as many as possible), and that unity and the point are thought by some to be first principles. Now, since the so-called genera are universal and indivisible (for there is no definition of them), some say the genera are elements, and more so than the differentia, because the genus is more universal; for where the differentia is present, the genus accompanies it, but where the genus is present, the differentia is

He writes, "'Elementary' propositions are those that are simple and elegant and have a variety of applications but do not rank as elements because the knowledge of them is not pertinent to the whole of the science . . ." As an example of an "elementary proposition" which is *not* an "element," Proclus cites the "theorem that the perpendiculars from the vertices of a triangle to the sides meet in a common point."[11] The point is, of course, the orthocenter of the triangle and when one thinks of later developments in geometry, such as the Euler line and Feuerbach circle, one wonders whether Proclus would still insist on the example, if he were writing today, that is, whether he would still consider it not an "element." Obviously, we do not mean by this to criticize Proclus for not knowing things that did not exist in his time, rather, we mean to emphasize that when Proclus says a theorem is not an "element" if "the knowledge of [it] is not pertinent to the whole of the science," one must understand that what is or is not an "element" is, therefore, *relative* to the whole. Hence, proper knowledge of the "elements" really demands knowledge of the whole. This is important, for we are asking why it is not strange that a perfectly competent geometer, and maybe more, such as Eudemus, should be interested in a course in the elements and even be asked to acquaint *only* those "worthy of sharing in such things" with it. What Apollonius has sent to Eudemus is not a set of simple propositions for beginners; the "course in the elements" he has sent is, in fact, the peculiar setting or framework for all of Apollonius's thought about conics.

The Initial Definitions

Book I of the Conica, the first book of Apollonius's "course in the elements," begins with nineteen definitions.[12] These definitions include that of the "conic surface (κωνικὴ ἐπιφάνεια)," "cone (κῶνος)," "diameter (διάμετρος)" of a plane curve, lines "drawn ordinatewise to the

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not always so. It is common to all the meanings that the element of each thing is the first component immanent in each" (1014a26–1014b16, W. D. Ross's translation). It is worth noting, that in the order of Aristotle's definitions in Book Δ the definition of "elements" comes after that of "cause" and before "nature," as if to say that "elements" are "causes" but not quite "natures."

[11] Morrow, pp. 59–60.

[12] These are arranged in eight groups in the Heiberg edition of the *Conica*. Taliaferro preserves Heiberg's arrangement in his translation.

diameter (τεταγμένως ἐπὶ τὴν διάμετρον κατῆχθαι)," "transverse diameter (διάμετρος πλαγία)," "upright diameter (διάμετρος ὀρθία)," "conjugate diameters (συζυγεῖς διάμετροι)," and "axes." Needless to say, all these definitions involve simple geometrical objects—points, lines, and planes—and, with perhaps one exception, simple geometrical relationships—intersecting, bisecting, being parallel. The definitions, for example, of the "diameter" of a curved line, "vertex" of a curved line, and lines "drawn ordinatewise to the diameter" are as follows: "Of any curved line which is in one plane I call that straight line the diameter which, drawn from the curved line, bisects all straight lines drawn to this curved line parallel to some straight line; and I call the end of that straight line [the diameter] situated on the curved line the vertex of the curved line, and I say that each of these parallels is drawn ordinatewise to the diameter."

As one can easily see, there is not (with the exception of "ordinatewise") a word here which one cannot find among those defined in Book I of Euclid's *Elements.* What a straight line is and how it differs from a curved line is clear to anyone who has read definitions 2 and 4 of Book I of the *Elements*; the vertex as "the end of that straight line situated on the curved line (τὸ πέρας τῆς εὐθείας τὸ πρὸς τῇ γραμμῇ)" is, in its language, certainly reminiscent of definition 3 in the *Elements*, "The extremities of a line are points (γραμμῆς δὲ πέρατα σημεῖα)."[13] The word "diameter" itself is familiar from definition 17 in Book I of the *Elements* where it is defined for the circle: "A *diameter* of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle (δίχα τέμνει τὸν κύκλον)." It is not entirely clear what Euclid means by saying the diameter "bisects the circle." He might mean that the diameter divides the circumference into to equal parts,[14] or, perhaps, the figure into two equal areas, a suggestion that finds some support in the definition of the "semicircle" which immediately follows that of the

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[13] Throughout, the translations from Euclid's *Elements* are by Sir Thomas Heath, *The Thirteen Books of Euclid's Elements.*

[14] This is what Thales is said to have proved according to Proclus: "They say that Thales was the first to demonstrate that the circle is bisected by the diameter (διχοτομεῖσθαι τὸν κύκλον ὑπὸ τῆς διαμέτρου), the cause of the bisection being the unimpeded passage of the straight line through the centre (ἡ τῆς εὐθείας ἀπαρέγκλιτος διὰ τοῦ κέντρου χώρησις)." (Translation by Ivor Thomas in *Selections Illustrating the History of Greek Mathematics*, I, 165). Euclid might well be referring to Thales when

"diameter." However, while Euclid does not prove this fact, he does prove in *Elem.*III.3, right at the start of his book about circles, that the diameter bisects all chords drawn to it perpendicularly. Hence, as the reader goes through the definitions of the *Conica* he can easily make the analogies[15] between "diameter of a circle" and "diameter of a curve," "lines drawn perpendicularly to the diameter" and "lines drawn ordinatewise to the diameter"; Apollonius will use the word "diameter" for any line *like* the diameter of a circle and will use "ordinatewise" *in the way* "perpendicularly" is used regarding chords drawn to a circle's diameter, leaving it open whether "ordinatewise" is identical to "perpendicularly" (indeed, we must wait until the *porism* following *Conica* I.7 before Apollonius points out explicitly ". . . that it is possible for some parallels to be bisected by the diameter FG and *not* [emphasis added] be perpendicular.").

Apollonius's geometrical outlook in the opening definitions is revealed not only by the appeal to simple geometrical objects and relations and to analogies with well-known geometrical facts, but it is also betrayed by Apollonius's *uneasiness* with notions which, though they may be logically necessary, lack visual intuitiveness. Consider the definitions corresponding to the "diameter," "vertex," and "ordinatewise" in the case of *two* curves. Apollonius writes: "Likewise of any two curved lines lying in one plane I call that straight line the transverse diameter which cuts the two curved lines and bisects all the straight lines drawn to either of the curved lines parallel to some straight line; and I call the ends of the diameter situated on the curved lines the vertices of the curved lines; and I call that straight line the upright diameter which, lying between the two curved lines, bisects all the straight lines intercepted between the curved lines and drawn parallel to some straight line; and I say that each of the

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he says the "diameter bisects the circle," although, of course, this goes no further in explaining what *Euclid* understands by "bisecting the circle."

[15] One should note that there are other, clearer, analogies between propositions in the *Conica* and propositions about circles in the *Elements.* For example, *Elem.*III.4 states: "If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another," while *Conica* II.26 states: "If in an ellipse or circumference of a circle two straight lines not through the center cut each other, then they do not bisect each other." Analogies can be made, too, between *Elem.* III.10 and *Conica* IV.25, 38, 43, 44, 46, 55, between *Elem.*III.1 and *Conica* II.45, *Elem.*III.30 and *Conica* II.47, *Elem.*III.35, 36 and *Conica* III.17,16 (also noted by Knorr in his *Ancient Tradition of Geometric Problems*, pp. 120–121). We shall have more to say about this in our chapter on conics and circles.

parallels is drawn ordinatewise to the diameter." It is clear that when Apollonius defined the transverse and upright diameters he had in mind the "opposite sections (τομαὶ ἀντικείμεναι)," which he was to discuss very soon. He is, of course, not explicit about this (he *could not* be, having not yet defined the opposite sections or any of the other conic sections) and has no diagram hinting at his specific intention, but Eutocius has a diagram (see fig. 2) illustrating these definitions and, although it aims at generality, one of the figures looks suspiciously like a double-branched hyperbola (*i.e.*, opposite sections). One could illustrate these definitions, however, quite as well with two non-intersecting circles (see fig. 3). If the centers of these circles are O and Q, then the line joining O and Q is the "transverse diameter," and A and B are the "vertices," as are D and C. EF, the perpendicular bisector of AB is, in this case, the "upright diameter." GH is drawn "ordinatewise" to the transverse diameter while HI is drawn "ordinatewise" to the upright diameter. This example highlights two points: 1) that there exist simple and well known geometrical objects through which the definitions may be understood, and which may have guided Apollonius in his creation, and 2) that the diameters spoken of here are of "*two* curves (δύο καμπύλων γραμμῶν)." This second point would suggest that Apollonius considers the opposite sections as two curves *tout court.* But, in fact, this is far from clear. Apollonius's ambivalence regarding the duality or singularity of the opposite sections runs all through the book.

Indeed, that the "transverse" and "upright diameters" refer to *two* curves in the definitions does not alone settle the issue, since Apollonius often uses the word "transverse (πλαγία)" to describe the diameter of a *single* curve. At the end of proposition I.13, for example, he refers to the diameter of the ellipse as the "transverse side,"[16] and in proposition I.14, the proposition introducing the opposite sections, he refers to the diameter of each of the hyperbolas—and Apollonius does treat the opposite sections here as *two* hyperbolas—as the "transverse side of the figure (τοῦ εἴδους ἡ πλαγια πλευρὰ)." In neither case,

[16] In this connection, one should note that in the Heiberg edition of Eutocius's commentary (pp. 200–201), at least, the text describing the transverse and upright diameters contains not only the diagram suggesting the opposite sections, which we mentioned above, but also one suggesting an ellipse. Perhaps Eutocius shared Apollonius's ambivalence regarding the doubleness or singleness of the opposite sections, but whatever the case, by juxtaposing the diagrams this way Eutocius was certainly reflecting Apollonius ambiguous usage of πλαγία.

however, does Apollonius refer, *explicitly*, to the "transverse *diameter*," leaving, thus, the whole matter rather ambiguous. Moreover, the *upright diameter*, whose definition makes sense *only* with regard to two curves, never appears in Book I and appears only briefly in Book II (propositions II.37,38). In Book I, one finds the "second diameter" defined after proposition I.16, but most often one finds the "conjugate diameter" where one would expect the "upright diameter." Even in the two propositions, II.37,38, where Apollonius does mention the upright diameter explicitly, he is quick to point out that it is conjugate to the transverse diameter. Indeed, Apollonius prefers the conjugate diameter to the upright diameter since the former is defined equally for one and two curved lines (καμπύλης γραμμῆς καὶ δύο καμπύλων γραμμῶν) and this allows him to "avoid the question" of the plural nature of the opposite sections. But, however ambiguous the text is on the issues of whether the opposite sections are one or two curves, we must face the fact that the *definition* of the transverse and upright diameters does refer unambiguously to two curves. The way to understand this is that while it may indeed make sense to treat the opposite sections as one curve, as a *visual* object the opposite sections are clearly two. Apollonius is, we think, working from this *visual* or *geometrical* intuition when he writes these definitions and it is the conflict with this intuition which is at the root of his uneasiness concerning the nature of the opposite sections and his avoidance of the clearly defined upright diameter; it seems, then, that Apollonius is unwilling to commit himself completely to the opposite sections as a single curve, if such a commitment cannot be grounded in some immediate visual intuition.

This tension we have described with regards to the opposite sections is also evident in Apollonius's treatment of the "cone" and the "conic surface." Apollonius defines these as follows:

> 1. If from a point a straight line is joined to the circumference of a circle which is not in the same plane with the point, and the line is produced in both directions, and if, with the point remaining fixed, the straight line being rotated about the circumference of the circle returns to the same place from which it began, then the generated surface composed of the two surfaces lying vertically opposite one another, each of which increases indefinitely as the generated line is produced indefinitely, I call a conic surface, and I call the fixed point the vertex, and the straight line drawn from the vertex to the center of the circle the axis.
>
> 2. And the figure (σχῆμα) contained by the circle and by the conic

> surface between the vertex and the circumference of the circle I call a cone, and the point which is also the vertex of the surface I call the vertex of the cone, and the straight line drawn from the vertex to the center of the circle the axis, and the circle the base of the cone.
>
> 3. I call right cones those having axes perpendicular to their bases, and oblique (σκαληνοὺς) those not having axes perpendicular to their bases.

As in the case of the diameter, there is a certain analogy between these and the corresponding definitions in Book XI of the *Elements*. In particular, the striking use of motion in Apollonius's definition of the conic surface has a clear precedent in Euclid's definition of the cone as well as in Euclid's definitions of the sphere and cylinder.[17] The latter are defined, respectively, as the figure generated by rotating a semicircle about its fixed diameter, and as the figure produced by rotating a rectangle about one of its sides. The cone and its parts are defined by Euclid in Book XI as follows:

> 18. When, one side of those about the right angle in a right angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved, the figure (σχῆμα) so comprehended is a *cone*.
>
> And, if the straight line which remains fixed be equal to the remaining side about the right angle which is carried round, the cone will be *right-angled*; if less, *obtuse-angled*; and if greater, *acute-angled*.
>
> 19. The *axis of the cone* is the straight line which remains fixed and about which the triangle is turned.
>
> 20. And the *base* is the circle described by the straight line which is carried round.

By definition, the axis of Euclid's cone is perpendicular to the base, so that Euclid's cone is what Apollonius calls a "right cone." By rotating a line, fixed at a point, along a fixed circle, however, Apollonius can produce this type of cone as well as "oblique cones," which cannot be produced, needless to say, by the rotation of any right-angled triangle. So, Apollonius's cone is more general than Euclid's cone and, thus, while there is an analogy between them, it is a limited one. What has *no* analogy in Euclid's definitions is the *conic surface*. The conic surface differs from the cone in two crucial respects. First, the conic surface is a "double" surface. This is why Apollonius often refers to the conic surface as the "vertically oppo-

[17] *Elements*, XI, definitions 14–23.

site *surfaces* [emphasis added]." Eutocius makes this plurality even more graphic by expanding on Apollonius's description and saying that the conic surface is ". . . a surface composed of *two surfaces touching one another* [emphasis added] at [the vertex] Δ (. . . ἐπιφάνειάν τινα, ἥτις σύγκειται ἐκ δύο ἐπιφάνειῶν ἁπτομένων ἀλλήλων κατὰ τὸ Δ)."[18] The cone is contained by the vertex and the circular base and so is a single surface. The "doubleness" of the conic surface is important since the opposite *sections* can be produced, as is seen in proposition I.14, only by slicing the conic surface. Second, the conic surface is *unbounded* (ἄπειρον). This, of course, is essential since neither a complete parabola nor a complete hyperbola could be produced from a bounded surface. (It is perhaps a trivial observation that even the ellipse or circle, which are bounded, are not realizable in a cone if the cutting plane cuts the base of the cone.) Hence, the conic sections, as sections, are only properly defined in relation to the conic surface. Yet in the whole first book, Apollonius refers explicitly to the conic surface in only four propositions: I.1, 2, 4, and 14. Needless to say, these do *not* include the propositions in which Apollonius defines the parabola, ellipse, and the single-branched hyperbola. We take this to be an active avoidance of the conic surface or, equivalently, a preference for the bounded, more visually appealing cone. In this connection, let us consider I.1 and I.4.

In proposition I.1, Apollonius shows that the straight line joining any point on the conic surface with the vertex is on the surface. The proof is very simple but revealing. Let A be the vertex of the conic surface and B a point on the surface (see fig. 4). Suppose AB is not on the surface. Let DE be the line generating the surface and EF the circle along which ED is moved. Then as DE is moved along EF it will at a certain point pass through B. But DE also passes through the fixed point A. So if ACB is not on the surface, we have two different straight lines passing through A and B, which is absurd.

One notices here, first of all, that not only does Apollonius refer to the conic surface, but also goes through the stages of its generation. This should not come as a surprise, since the definition of the conic surface is in terms of its mode of production.[19] What the

[18] Eutocius, *In Conica*, p. 188, our translation.

[19] This is, of course, true also of Euclid's definition; to show *what* a cone is, Apollonius and Euclid show *how it is produced.* We should add that Euclid defines the pyramid, sphere and cylinder also by describing their mode of production. Such

construction implies is that the point B can be as far as one pleases from the vertex A, and that, above and below A. Yet, instead of the two cases that this suggests, one sees three. The diagram shows three cases according to whether the point B is in the bounded region of the surface between the vertex A and the circle EF, that is, on the cone, or in the unbounded region of the surface below EF, or in the unbounded upper surface. The distinction, when B is on the lower surface, between B being below the base circle or B being above the base circle and below the vertex seems quite unnecessary; it certainly does not play any part in the logic of the proof. It must reflect, therefore, only Apollonius's need to introduce into the discussion the distinction between the bounded cone and the unbounded conic surface. Thus, in the course of proving I.1, Apollonius manages to bring out at the same time *precisely* the two qualities of the conic surface mentioned above, namely, its doubleness and its unboundedness. It is perhaps for this reason, too, that the greater part of Eutocius's remarks on I.1 focus on the three cases presented by the diagram. This tripartite diagram is a kind of motif for the conic surface. Indeed, it appears again in proposition I.2, in which Apollonius shows that if any two points lie on one of the vertically opposite surfaces, then the line joining them falls inside the surface, provided it does not verge to the vertex, and, again, in I.4, which we shall now discuss.[20]

Proposition I.4 states: "If either one of the vertically opposite surfaces is cut by some plane parallel to the circle along which the straight line generating the surface is moved, the plane cut off within the surface will be a circle having its center on the axis, and the figure contained by the circle and the conic surface intercepted by the cutting plane on the side of the vertex will be a cone." The

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definitions are in keeping with Aristotle's description of a proper definition in his discussion on the subject in the *Posterior Analytics* (90aff.). Aristotle tells us there that "definition exhibits the essence (τί ἐστι δηλοî)," but in order to show *what a thing is*, that is, to show its essence (τὸ τί ἔστιν), one must show its cause, that *through which is it is* (διὰ τί ἔστιν). Putting this in terms of the present context, to show *what* a cone is would, for Aristotle (and for Euclid and Apollonius, so it seems), be to show that *through which* it is what it is, that is, its construction.

[20] In the list of propositions employing the "conic surface," we did not include I.6 because it speaks of "a point on the surface of a cone (τι σημεῖον ἐπὶ τῆς τοῦ κώνου ἐπιφανείας)" and not of the "conic surface (κωνικὴ ἐπιφάνεια)" *explicitly*. However, the latter is evidently implied, for the tripartite diagram of I.1, 2, and 4, the motif of the conic surface, appears also in I.6.

proposition tells us, thus, two things: 1) that a section parallel to the generator circle is itself a circle and 2) that this circular section can be thought of as the base of a cone. The second point, which is of interest to us in the present discussion, makes this proposition a kind of bridge between the conic surface and the cone. Here is Apollonius's proof of the theorem:

> Let there be a conic surface whose vertex is the point A (see fig. 5) and whose circle along which the straight line generating the surface is moved is BC; and let it be cut by some plane parallel to the circle BC, and let it make on the surface as a section the line DE.
>
> I say that the line DE is a circle having its center on the axis.
>
> For let the point F be taken as the center of the circle BC, and let AF be joined.
>
> Therefore AF is the axis and meets the cutting plane. Let it meet it at the point G, and let some plane be produced through AF. Then the section will be the triangle ABC. And since the points D, G, E are points in the cutting plane, and are also in the plane of the triangle ABC, therefore DGE is a straight line.
>
> Then let some point H be taken on the line DE, and let AH be joined and produced. Then it falls on the circumference BC. Let it meet it at K, and let GH and FK be joined. And since two parallel planes DE and BC are cut by a plane ABC, their common sections are parallel. Therefore, the straight line DE is parallel to the straight line BC. Then for the same reason the straight line GH is also parallel to the straight line KF. Therefore,
>
> $$FA:AG::FB:DG::FC:GE::FK:GH.$$
>
> And,
>
> $$BF = KF = FC.$$
>
> Therefore also,
>
> $$DG = GH = GE.$$
>
> Then likewise we could show also that all the straight lines falling from the point G on the line DE are equal to each other.
>
> Therefore the line DE is a circle having its center on the axis.
>
> And it is evident that the figure contained by the circle DE and the conic surface cut off by it on the side of the point A is a cone.
>
> And it is therewith proved that the common section of the cutting plane and of the axial triangle . . . is a diameter of the circle.

The bulk of this proof shows that the section described, parallel to the base circle, is itself a circle; that Apollonius has produced herewith a cone from the conic surface is summed up only in the phrase "it is evident (καὶ φανερόν)." Why is this "evident," and if it is so

evident why does Apollonius need to state it, even in the enunciation? First, why is it evident that the figure contained by DE, A, and the conic surface is a cone? The line AH joins the vertex with a point on the surface therefore it lies entirely on the surface by I.1. Extended, this line must fall on the circumference of BC as stated in the proof. So one may see the line AH as the line generating the conic surface. As AH moves along the circumference BC it also moves about the line DE. With this, and having proved that DE is a circle, one can then say it is evident that we have a cone with DE as its base since the line AH moving along the circle DE *generates a conic surface.* The three diagrams given in the body of the proof, whose main significance is clear from our discussion above, may be thought of as three "snapshots" of the generation of the surface just described. In each, one sees the line AH joining A and the circumference of the circles BC and DE emphasizing the symmetry of their roles. Indeed, the first two diagrams are indistinguishable except for the labelling of the upper and lower circle, while in the third diagram the configuration one sees above the vertex A is indistinguishable from that below A except, again, for the labelling. These arguments are not proofs but *visualizations* and they are convincing to the extent that one can *see* in the proof and in the diagram the genesis of the conic surface and the figure of the cone as they are described in Apollonius's definitions. The visualizations have weight because DE is *proved* to be a circle. Since DE being a circle is the *only* thing proved in the proposition and arguments such as those we have just given are unstated, one must conclude that when Apollonius says "*it is evident* [emphasis added] that the figure contained by the circle DE and the conic surface cut off by it on the side of the point A is a cone," he means just that, namely, that what is stated can be *seen* to be so. Incidentally, we remind the reader that the word "evident (φανερόν)" means literally, "open to sight."[21]

[21] The visual character of Apollonius's mathematics emphasized here is restricted to the *approach* to mathematical problems and concepts, to a kind of *visual bias* with which the Hellenistic mathematician handled these things. This is not the same thing as saying that Hellenistic mathematics was a visual mathematics. (See Árpád Szabó, "The Transformation of Mathematics into Deductive Science and the Beginnings of Its Foundation on Definitions and Axioms," in *Scripta Mathematica*, vol. XXVII, No. 1, pp. 34–42, for an elaboration of what Szabó calls the "anti-illustrative tendency" in Greek science.) The Greek mathematician of Apollonius's time clearly wanted to anchor his ideas in *logos*, in precise words without recourse

The cone's being "open to sight" is perhaps the reason why Apollonius prefers the cone to the conic surface, as we observed above, and why he emphasizes both in the enunciation and proof of I.4 that by sectioning a conic surface parallel to its generating circle one produces a cone, even though this is "evident." Indeed, one of the differences between Apollonius's description of the cone and that of the conic surface is that the cone, unlike the conic surface, is called a "figure (σχῆμα)." In Euclid's definition, too, the cone is a "figure," as is the pyramid, prism, sphere, cylinder, cube, octahedron, icosahedron, and dodecahedron.[22] Euclid defines "figure" in the first book of the *Elements* as "that which is contained by any boundary or boundaries (τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον)," where a "boundary is that which is an extremity (πέρας) of anything."[23] This is comparable to the "geometrical" definition Socrates gives in the *Meno*: "I say that shape (σχῆμα) is that in which a solid terminates (περαίνει), or more briefly, it is the limit (πέρας) of a solid."[24] Earlier, Socrates gives a slightly different definition which he seems to prefer. He says to Meno, "Well now, let's try to tell you what shape (σχῆμα) is. See if you accept this definition. Let us define it as the only thing which always accompanies color. Does that satisfy you, or do you want it in some other way? I should be content if your definition of virtue were on similar lines."[25] Thus, the cone as a figure is bounded, it has limits (πέρα), as opposed to the surface which, as we noted above, is unbounded (ἄπειρον). The cone can, thus, be grasped whole by the eye, while the conic surface

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to imprecise pictures (in this connection, one thinks of the Stranger's remark in Plato's *Statesman*, ". . . that a definition couched in words is a better description of a living creature than a drawing or any model of it can be—a better description, I mean, for those capable of following such a definition . . ." (277c)). Hence, no diagram ever accompanies a definition or an enunciation. On the other hand, the parts of a proposition following the enunciation, the "setting out (ἔκθεσις)," "specification (διορισμός)," and "construction (κατασκευή)," present *always* a specific case given together with a diagram, with proof which follows referring to *this* case and *this* diagram. It is in this part of Greek mathematics that the visual character of the ideas, the possibility of their being represented in a diagram, is crucial. Interesting discussions of diagrams and the importance of diagrams in the proof of propositions can be found in Wilbur Knorr, *The Evolution of the Euclidean Elements*, pp. 69–74, and in Reviel Netz, "Greek Diagrams: Their Use and Their Meaning."

22 *Elem.*XI, def. 12–14, 21, 25–28.

23 *Elem.*I, def. 13, 14.

24 Plato, *Meno*, 76a4–7, trans. By W. K. C. Guthrie.

25 *Ibid.*, 75b8–c1. For an enlightening discussion of the entire passage, see Jacob Klein, *A Commentary on Plato's Meno*, pp. 55–70.

as a whole can never be. In this sense, we reaffirm that the cone is a visual object in a way that the conic surface is not. Indeed Socrates' preferred definition of figure, which links figure inseparably with color, links figure inseparably, therefore, with vision. Is it not for this reason, too, that *we* call diagrams, which are obviously the most visual elements of a text, "figures"? So it is as a figure, as a thing which the eye can grasp whole, that Apollonius prefers using the cone in the text rather than the conic surface.

Despite his preference for the cone, however, Apollonius understands the necessity of the conic surface for properly defining the conic curves, and so it is imperative that there be a proposition linking the conic surfaces and cone in such a way that it be possible to speak about these sections in terms of the cone. This is what proposition I.4 accomplishes in showing how from a given conic surface one may derive a cone having its base as far as one pleases from the vertex and on whatever side. Indeed, in I.5, Apollonius proves that when an oblique cone is cut subcontrariwise[26] the section produced is a circle, but he does *not* point out here that the figure contained by the surface of the cone, the subcontrary circle, and the vertex is a cone. This can only mean that Apollonius is not interested in a theoretical query concerning the number of ways a cone may be derived from a conic surface, only that there *be some* way. For having some way to move back and forth between the conic surface and the cone allows Apollonius to avoid the conic surface altogether and talk only about the cone.

To illustrate further what we have been saying, let us consider proposition I.8. It states that, "If a cone is cut by a plane through its axis, and is cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the resulting section on the surface is either parallel to one of the sides of the triangle or meets one of them beyond the vertex of the cone, and the surface of the cone and the cutting plane are produced indefinitely, then the section will also increase indefinitely, and some straight line drawn from the section of the

[26] By "subcontrariwise (ὑπεναντίως)," Apollonius means that the common section of the cutting plane and the axial triangle, forms with the sides of the axial triangle angles equal to those at the base of the axial triangle but on opposite sides as in fig. 6 where angle G equals angle C and angle K equals angle B. Mugler cites Apollonius only as using this term in this way (*Dictionaire Historique*, pp. 434–435).

cone parallel to the straight line in the base of the cone will cut off from the diameter on the side of the vertex a straight line equal to any given straight line." The main part of Apollonius's proof is as follows:

> Let there be a cone whose vertex is the point A and whose base is the circle BC (see fig. 7), and let it be cut by a plane through its axis, and let it make as a section the triangle ABC. And let it be cut also by another plane cutting the circle BC is a straight line DE perpendicular to the straight line BC, and let it make as a section on the surface the line DFE. And let the diameter FG of the section DFE be either parallel to the straight line AC or on being produced meet it beyond the point A.
>
> I say that, if both the surface of the cone and the cutting plane are produced indefinitely, the section DFE also will increase indefinitely.
>
> For let both the surface of the cone and the cutting plane be produced. Then it is evident that also the straight lines AB, AC, FG will be therewith produced. Since the straight line FG is either parallel to AC or produced meets it beyond the point A, therefore the straight lines FG and AC on being produced in the direction of C and G will never meet. Then let them be produced and let some point H be taken at random on the straight line FG, and let the straight line KHL be drawn through H parallel to the straight line BC and MHN parallel to DE. Therefore the plane through KL and MN is parallel to the plane through BC and DE. Therefore the plane KLMN is a circle.
>
> And since the points D, E, M, N, are in the cutting plane and also on the surface of the cone, therefore they are on the common section. Therefore the section DFE has increased to the points M and N. Therefore, with the surface of the cone and the cutting plane increased to the circle KLMN, the section DFE has also increased to the points M and N. Then likewise we could show also, that if the surface of the cone and the cutting plane are extended the section MDFEN will also increase indefinitely . . .

After the enunciation, the so called "setting out (ἔκθεσις)" prescribes a cone, a plane through the axis, and another plane cutting the cone to produce the section. All this is carried out according to propositions I,3,6,7. And since these are propositions in which the given is a cone, one might claim that herein is the dependence on the cone. However, these propositions might as well have been expressed in terms of the conic surface, so such a claim would, indeed, be very weak. Perhaps more significant is the construction after the proof itself begins. Apollonius tells us that the surface of the cone is to be produced. He does not tell us how it is to be produced but he does say later that the surface of the cone has increased to the circle

KLMN. Apollonius, at this point, is using proposition I.4. However, here he *increases* the surface of the cone *to* a plane whereas in I.4 an *already produced* conic surface is cut by a plane. The effect of this is to force us to imagine a cone being increased from one base to another, or, rather, a sequence of cones with ever increasing bases. Indeed, this is precisely what we see in the diagram; it shows no conic surface, only cones. So not only does Apollonius prefer the visually appealing cone over the not-so-easily pictured conic surface, but he actively avoids the latter, using the cone when it would be natural and, perhaps, more correct to use the conic surface.

Let us review the argument. Although the conic surface is produced by a visually appealing procedure, it is not itself a visually appealing object. In particular, it is not a bounded figure (σχῆμα), a shape, the sort of thing portrayable wholly by a diagram. The cone is. Apollonius's concern with this point induces him to use the cone rather than the conic surface whenever possible, even when using the conic surface would have been more appropriate. His uneasiness with the conic surface, as with the opposite sections, points to a mode of thinking in which, at least to some extent, visibility is a measure of understandability.

The Definitions of the Conic Sections and the Symptomata

The most conspicuous omission from the opening definitions is, of course, that of the definitions of the parabola, hyperbola, ellipse, and opposite sections. Apollonius delays the definitions of these curves until after the first ten propositions and then defines them in the course of propositions I.11–14. The reason for this delay becomes clear when we consider what the ten preliminary propositions say and how they fit together.

To begin the discussion, however, let us return to the difference between Apollonius's and Euclid's cone. In Euclid's cone, we recall, the axis is perpendicular to the base by definition, for the cone is produced by turning a right triangle about one of its legs. Euclid can then divide all cones into three types according to whether the fixed leg, the leg forming the axis, is equal to, shorter, or longer than the leg carried around. Since these three situations imply that the vertex angle of the cone will be right, obtuse, or acute, respectively, Euclid calls the types of cones "right-angled (ὀρθογώνιος)," "obtuse-angled (ἀμβλυγώνιος)," and "acute-angled (ὀξυγώνιος)." Heath

points out that "This distinction is quite unnecessary for Euclid's purpose and it is not used by him in Book XII; it is no doubt a relic of the method, still in use in Euclid's time, by which the earlier Greek geometers produced conic sections, namely, by cutting right cones only by sections always perpendicular to an edge."[27] This is the tradition of Menaechmus[28] and its significance lies in the fact that since the manner of cutting the cone is fixed, the sort of curve one obtains depends *only* on the *type* of cone. Accordingly, the section of a right-angled cone is a parabola, the section of an obtuse-angled cone is a hyperbola, and of an acute-angled cone, an ellipse. It is obvious, moreover, that in a given right cone, cut in this specified way, all the ellipses and hyperbolas obtained are similar.[29] Apollonius's cone, however, is, in general, an *oblique cone*, and with an oblique cone the situation is different. While it is true that a section made perpendicular to an edge of a given oblique cone will produce always a parabola, or always an ellipse, or always a hyperbola,[30] it is *not*

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[27] Heath, *Elements*, III,270.

[28] Wilbur Knorr in his "Observations on the Early History of the Conics" has argued that Menaechmus had no theory of conic sections that "The notion that around the middle of the fourth century Menaechmus initiated a theory of conics is purely inferential and fits poorly with what we know of the interests and techniques of geometers at that time" (pp. 18–19). Of course, the information we have about this early period is scant, so that, as Knorr concedes, "[his] view rests on inference, as the standard view does . . ." (p. 19). However, even if Knorr is right, the relevant point in the present study is that Apollonius's mode of sectioning the cone was an innovation over an older method and this we have on the authority of Eutocius who says quite explicitly that ". . . the ancients investigated the so-called *section of a right-angled cone* (ὀρθογωνίου κώνου τομὴν) in a right angled cone only, cutting it by a plane perpendicular to one side of the cone, and they demonstrated the *section of an obtuse-angled cone* (ἀμβλυγωνίου κώνου τομὴν) in an obtuse-angled cone and the *section of an acute-angled cone* (ὀξυγωνίου [κώνου τομὴν]) in an acute-angled cone, in the cases of all the cones drawing the planes in the same way perpendicularly to one side of the cone; hence, it is clear, the ancient names of the curves. But later Apollonius of Perga proved generally that all the sections can be obtained in any cone, whether right or scalene, according to different relations of the plane to the cone" (trans. By Ivor *Thomas in Greek Mathematical Works*, II, p. 279).

[29] The equality and similarity of conic sections are defined and discussed fully by Apollonius in Book VI of the *Conica*. We shall look at this book in detail in a later chapter.

[30] With the sectioning procedure so determined, whether a given section is a parabola, hyperbola, or ellipse depends only on whether the angle at the vertex of the axial triangle in the plane perpendicular to the cutting plane be right, obtuse, or acute. However, in a given oblique cone, this angle is *always* right or *always* obtuse or *always* acute. This is clear, since if ABC is an axial triangle (see fig. 8), then BC is the diameter of the cone's base and AD, which is obviously a median of triangle ABC, is the cone's axis. These lines are fixed in length. So if the axis

true that all these ellipses are similar or all the hyperbolas are similar.[31] Hence, Apollonius's approach, by contrast, is to take a *given* cone and *section it in various ways*—an approach which carries with it the great discovery that all three types of conic sections can be obtained from *one* given cone. Since sectioning the cone is no longer rigidly prescribed, as it was in the earlier theory, Apollonius must initially determine *general* properties of sections and sectioning, and he must do this before speaking about any particular section, namely, about the sections producing the parabola, ellipse, and the rest. This is the purpose of the first ten propositions. Together, these preliminary propositions form a *mini-treatise* on the cone and the general properties of sections in a cone.[32] We have already examined some of these propositions in order to see more clearly how Apollonius understands the cone and conic surface. Let us now re-examine them, briefly, to see how they pave the way for Apollonius's definitions of the conic sections in propositions I.11–14.

Propositions I.1 and I.2 are at once general propositions about the conic surface and also important lemmas for the remaining eight propositions. They are both used implicitly in proposition I.3, which defines the first elementary section, the triangle. Apollonius's enunciation and proof of I.3 are as follows: "If a cone is cut by a plane through the vertex, the section is a triangle.

> Let there be a cone whose vertex is the point A and whose base is the circle BC (see fig. 9); and let it be cut by some plane through the point A; and let it make, as sections, lines AB and AC on the surface, and the straight line BC in the base.
>
> I say that ABC is a triangle.
>
> For since the line joined from A to B is the common section of the cutting plane and of the surface of the cone, therefore AB is a straight line. And likewise also AC. And BC is also a straight line. Therefore ABC is a triangle.

That BC is a straight line, we know from *Elem.*XI.3 since the base of a cone is a plane; that it is in the base and not just in the plane of the base we know from *Conica* I.2. Apollonius does not spell out

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of the cone is equal to the radius of its base, angle A is right in every axial triangle; if the axis is shorter than the radius, then angle A is obtuse in every axial triangle; and if the axis is shorter than the radius of the base, then angle A is acute.

[31] *All* parabolas are similar, as Apollonius proves in proposition VI.11.

[32] Thus Eutocius stresses the integrity of these ten propositions, saying "Χρὴ ἐπιστῆσαι, ὅτι τὰ ι′ ταῦτα θεωρήματα ἀλλήλων ἔχονται" (*In Conica*, p. 214).

why AB and AC are straight lines for the argument would be essentially that of I.1. As for I.3 itself, it is used as a lemma, but, at the same time, it shows us the first general section. For when Apollonius uses the proposition, he always uses the axial triangle, the special case one obtains when the cutting plane contains the axis of the cone. However, proposition I.3 does not specify this. Thus, the fact that *any* section through the vertex is a triangle remains a theoretical result important only in a general discussion of the sections in a cone.

The second elementary section, the circle, is given in propositions I.4 and I.5. As discussed above, proposition I.4 shows that a section made parallel to the generator circle produces a circle, and proposition I.5 shows that there is another distinct section producing a circle, the "subcontrary section." In proposition I.4 the cutting plane cuts off from the axial triangle another similar and similarly placed triangle, while in I.5, the cutting plane also cuts off from the axial triangle a similar triangle but one oppositely placed. Proposition I.5 is a stunning and surprising result, but it is, perhaps, the sort of proposition which Proclus calls "elementary," rather than "elemental," that is, as quoted above, one of those propositions which are "simple and elegant and have a variety of applications but do not rank as elements because the knowledge of them is not pertinent to the whole of the science."[33] Indeed, the subcontrary section never appears again in the *Conica.* On the other hand, I.5 is the first result unique to the oblique cone. For since the axial triangle in the right cone is always isosceles, the subcontrary section is indistinguishable from the section parallel to the base, and, therefore, in a right cone there can be *only one* way to obtain a circular section. So, with I.5, the reader sees quite clearly that sectioning an oblique cone is by no means

[33] Friedlein, p. 72, Morrow p. 59. It is true, that proposition I.5 provides a simple way to show that circles are preserved under stereographic projection. But there is no evidence this application was anywhere near Apollonius's thoughts. Heath confirms this, saying, "The fact that the method of stereographic projection is so easily connected with the property of subcontrary sections of oblique circular cones has led to the conjecture that Apollonius was the discoverer of the method. But Ptolemy makes no mention of Apollonius, and all that we know is that Synesius of Cyrene (a pupil of Hypatia, and born about A.D. 365–370) attributes the discovery of the method and its application to Hipparchus . . ." (*History,* II,292–293). The claim that I.5 shows that Apollonius came close to discovering inversive geometry (cf. Mugler, *Dictionaire Historique,* p. 434) cannot be taken seriously; it is an example of pure Whig historiography.

a trivial procedure and it demands, therefore, careful development.

Proposition I.9 would appear to complete the sequence I.4,5. For it states "If a cone is cut by a plane meeting both sides of the axial triangle, and neither parallel to the base nor situated subcontrariwise, then the section will not be a circle," so that, together, the three propositions say that a section will be a circle *if and only if* it is made according to I.4 or I.5. However, this raises the obvious question why is I.9 not I.6 in Apollonius's text? One reason is that it depends on proposition I.7. This explains why it cannot appear immediately after I.5, but it does not explain why it does not appear immediately after I.7. Since it does not depend on proposition I.8, Apollonius apparently wanted to stress the couple I.8,9 rather than the triple I.4,5,9, or, put another way, Apollonius saw I.9 as the complement of I.8 rather than the completion of I.4,5. This is so in the sense that while in I.8 the cutting plane either is parallel to a side of the axial triangle or meets a side of the axial triangle beyond the vertex, in I.9 the cutting plane cuts both sides of the axial triangle (by "sides" Apollonius means the sides having as a common vertex, the vertex of the cone) on the same side of the vertex. And these sections represent, in general, the outcome of the three ways of cutting a cone. One notes, also, that these three sections correspond to the perpendicular section of the three right cones, the right-angled, the obtuse-angled, and the acute-angled cone, respectively.[34] In this way, I.8,9 present a more coherent preparation for I.11–13 in which the parabola, hyperbola, and ellipse are defined, then would the sequence I.4,5,9. The correspondence between I.8,9 and I.11–13 is seen also in the nearly identical hypotheses of I.8 and I.11,12 and of I.9 and I.13. Thus, proposition I.8 begins "If a cone is cut by a plane through its axis, and is cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the resulting section on the surface is either parallel to one of the sides of the triangle or meets one of them beyond the vertex . . ." matching perfectly in Greek the beginning of the enunciation for proposition I.11, "If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the

[34] This is why, we think, Eutocius brings up the cutting schemes relative to the axial triangle for the various sections while reviewing the contents of I.9 (*In Conica*, pp. 214–216).

axial triangle and if further the diameter of the section is parallel to one side of the axial . . .," and proposition I.12, "If a cone is cut by a plane through its axis, and also by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if the diameter of the section produced meets one side of the axial triangle beyond the vertex of the cone . . ." Complementing these, proposition I.9 begins, "If a cone is cut by a plane meeting both sides of the axial triangle, and neither parallel to the base nor situated subcontrariwise . . .," matching I.13, "If a cone is cut by a plane through its axis, and is also cut by another plane on the one hand meeting both sides of the axial triangle, and on the other extended neither parallel to the base nor subcontrariwise . . ."

The use of I.7, which, together with the Porism following it, establishes the existence of a diameter in any section of a cone, is conspicuous in the propositions just cited. We shall discuss I.7 later, but for now suffice it to say that, besides establishing the existence of the diameter, this proposition presents the diameter in such a way that it may be used to locate the section within the cone, for it is by the relative positions of the diameter and the sides of the same axial triangle used in the construction of the diameter that Apollonius specifies the section producing a parabola, ellipse, and hyperbola in I.11–13.

To summarize, the inclusion of this *mini-treatise* on the cone and the general properties of its sections at the very beginning of the *Conica* shows that Apollonius did not think the conic curves as definable *before* developing sufficiently the geometry of the general section of the general cone. It is crucial to see in these preliminary propositions the importance Apollonius places on understanding the nature of sectioning. The cone and the general geometric character of its sections are the first elements of the conic curves. The cone is the fundamental mathematical object of the *Conica.* All other objects, including the sections, are derived from it. The cone is not a mere likeness of something more basic, a sort of mathematical *simulacrum* behind which lurks a simpler and more general way of deriving the curves. The definition of the curves for Apollonius is inconceivable without the cone. Therefore, any apt consideration of propositions I.11–14, and of Apollonius's understanding of the parabola, hyperbola, and ellipse embodied in them, must be done against the background of the geometrical character of the cone and its sections, revealed in propositions I.1–10. In this light, let us look at I.11.

The enunciation of proposition I.11 is as follows: "If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if further the diameter of the section is parallel to one side of the axial triangle, then any straight line which is drawn from the section of the cone to its diameter parallel to the common section of the cutting plane and of the cone's base, will equal in square the rectangle contained by the straight line cut off by it on the diameter beginning from the section's vertex and by another straight line which has the ratio to the straight line between the angle of the cone and the vertex of the section that the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle. And let such a section be called a parabola."

This long enunciation is both an enunciation of a proposition and also a definition, ending, as it does, with the words "And let such a section be called a parabola (καλείσθω δὲ ἡ τοιαύτη τομὴ παραβολή)." What, however, do the words "such a section" refer to? There are two possibilities: 1) they might refer to the section possessing the "fundamental property" given, the *symptoma* (σύμπτωμα), namely, that if K is some point on the section (see fig. 10) and KL is drawn parallel to the common section DE of the cutting plane and the base of the cone, and meeting the diameter FL at L then,

sq.KL = rect.HF,FL

where HF is a fixed line, determined by FA, and the three sides of the axial triangle (HF:FA::sq.BC:rect.AB,AC), or 2) "such a section" might refer to the section produced so that its diameter FL is parallel to one side AC of the axial triangle ABC. Which of these does Apollonius consider as the definition of the parabola?

Two arguments support 1) as being the definition. The first is that the *names* of the curves given by Apollonius correspond to the *symptomata*, and the second is that the *symptomata* provide a means of *identification* and *distinction* of the conic sections. As to the first, it is true that the names, "parabola (παραβολή)," "hyperbola (ὑπερβολή)," and "ellipse (ἔλλειψις)" derive from the language of application of areas which is, of course, the language, too, of the *symptomata* and it is clear that Apollonius wanted to connect these curves to the tradition of application of areas. This we have on the authority of Proclus who writes, "These things are ancient, says Eudemus, being

the discoveries of the Muse of the Pythagoreans, I mean the application of areas (παραβολὴ τῶν χωρίων), their exceeding (ὑπερβολή) and their falling short (ἔλλειψις). From these men the more recent geometers took the names that they gave to the so-called conic lines, calling one of these the *parabola*, one the *hyperbola* and one the *ellipse*, inasmuch as those god-like men of old saw the things signified by these names in the construction, in a plane, of areas upon a finite straight line."[35] But however important this may be, and the association with application of areas *is* important, such an association does not prove that Apollonius sees in the application of areas the "whatness" of the conic curves, that is, that the property of the curves connected with the application of areas is enough to *define* the curves. The problem with the second argument, that the *symptomata* provide a means of *identification* and *distinction* of the conic sections is that while the *symptoma* of, say, the parabola distinguishes it from all other *conic sections* it is not clear that it also distinguishes it from all other *curves*, not necessarily produced by sectioning a cone. Thus, Apollonius says "And let such a *section* (τοιαύτη τομὴ) [emphasis added] be called a parabola," and not "And let such a *curve* (τοιαύτη γραμμὴ) be called a parabola." Indeed, KL, HF, and FL, in terms of which

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[35] Translation by Ivor Thomas in *Greek Mathematical Works*, I,186–189. Clearly, the Eudemus spoken of here, presumably Eudemus of Rhodes, is not the Eudemus to whom Apollonius addressed his prefatory letter to Book I. For the former, being a student of Aristotle, must have been born about a century earlier than Eudemus of Pergamon.

This is the appropriate place to point out that it is not a completely settled issue whether Apollonius invented the terms "parabola," "ellipse," and "hyperbola," though it is highly probable that this is, indeed, the case. On the affirmative side, we have Pappus who tells us *explicitly* that Apollonius, ever so aware that any conic section could be derived from any cone, ". . . called the (section) of an acute angled (cone) 'ellipse', that of a right-angled (cone) parabola', and of an obtuse-angled 'hyperbola', each from a certain property of its own (Hultsch, p. 674, trans. by Jones, p. 116). Eutocius only says ". . . the [section] of a right-angled [cone] now called 'parabola' . . ." etc. (Heiberg, pp. 168–170), but when he says "now" he clearly implies "since Apollonius." But the strongest evidence of the Apollonian provenance of these names, however, is that in the prefatory letter to the first book, Apollonius never mentions the names "parabola," "hyperbola," and "ellipse," as if known and refers only to ". . . the three sections and the opposite branches . . ." At the same time, as we have said, the enunciations to propositions I.11–14 each end with the words "And let such a section be called . . ." etc., as if Apollonius is introducing these names for the first time. On the negative side, there are at least two occasions where Archimedes uses the terms "parabola" and "ellipse." These two instances are, respectively, in the *Method* (Heiberg II, pp. 436, 498) and in *Conoids and Spheroids* (Heiberg I, pp. 292, 298, 300). However, these are, most likely, later interpolations (see Toomer, *Diocles: On Burning Mirrors*, p. 4, note 3).

the *symptoma* is defined, are determined by the way Apollonius sections the given cone, so that the *symptoma* is subordinate to Apollonius's sectioning procedure. Since the second possibility above refers *directly* to this sectioning procedure, it would appear to be the most reasonable of the two.

Three further arguments support the conclusion that, for Apollonius, what determines, and thus defines, the conic curves is, first of all, the way the cone is sectioned, and not the *symptoma.*

1) The most obvious argument is that what is *proved* in I.11 is the *symptoma*; a definition is not something that needs proof, rather, in Aristotle's words, a definition "only need[s] to be understood."[36] And, recalling our discussion of the definition of the cone and conic surface, what understanding the definition of the parabola means for Apollonius is that its genesis has been presented to the eye. Indeed, what is *shown* in I.11 is the way in which the cone is to be sectioned whereas what is *deduced* is the *symptoma* of the section thus obtained.

2) If it is its *symptoma* which is the "essence (τὸ τί ἔστι)," the definition, of the parabola, it would be hard to understand Apollonius's careful development of the properties of sections as such, the "minitreatise" comprising the first ten propositions. In fact, it would be hard even to understand the need for the general oblique cone. For if the *symptoma* is the *definiens* of the parabola then the cone becomes merely a derivative way of representing the parabola, of displaying its geometrical occurrence. But for this *any* cone would do. In particular, if this were the only purpose of the cone, the "Euclidean" cone would be sufficient.

3) In the prefatory letter to Book I, Apollonius clearly separates the genesis of the conic curves, and thus their definitions, from their *symptomata.* For he writes, "The first book contains the generation of the three sections and of the opposite branches, *and* [emphasis added] the principal properties in them (καὶ τὰ ἐν αὐταῖς ἀρχικὰ συμπτώματα) . . ." The *symptoma* is a property of the parabola; it is separable from the parabola when the curve is generated and, thus, cannot be its definition. One sees this separation very clearly in proposition I.52 where Apollonius "constructs" the parabola. The enunciation is as follows: "Given a straight line in a plane bounded at one point, to find in the plane the section of a cone called parabola,

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[36] *Post. Anal.*, 76b37.

whose diameter is the given straight line, and whose vertex is the end of the straight line, and where whatever straight line is dropped from the section to the diameter at a given angle, will equal in square the rectangle contained by the straight line cut off by it from the vertex of the section and by some other given straight line." Let us consider Apollonius's construction when the given angle is right (the completion of the proposition for the general case is, in fact, contained in proposition I.53). To start, we are given two straight lines AB and CD in which "AB [is] given in position and bounded at the point A" while CD is "given in magnitude" (see fig. 11). We are required to find a parabola in the plane of reference for which CD will be the upright side (ὀρθία) and AB, the diameter (in fact, the *axis*, as Apollonius points out, since in this special case the lines drawn ordinatewise to AB will be at right angles to AB). Next, Apollonius builds a cone which will have the desired parabola as a section. He says,

> let AB be produced to E, and let CG be taken as the fourth part of CD, and let
>
> $$EA > CG$$
>
> and let
>
> $$CD:H::H:EA$$
>
> Therefore
>
> $$CD:EA::sq.H:sq.EA$$
>
> and
>
> $$CD < 4EA;$$
>
> therefore also,
>
> $$sq.H < 4sq.EA.$$
>
> Therefore
>
> $$H < 2EA;$$

Hence, one can construct an isosceles triangle EAF having its base equal to H and its sides equal to EA. Apollonius constructs this triangle perpendicular to the plane of reference and then another, AFK, whose base, AK, is parallel to FE and whose side FK is parallel to EA. From this, he produced a right cone of which AFK is an axial triangle: "[L]et a cone be conceived whose vertex is the point F and whose base is the circle about diameter KA, at right

angles to the plane through AFK. Then the cone will be a right cone, for AF = FK."

At this stage, Apollonius shows that the plane of reference cuts the cone precisely as prescribed in proposition I.11. He writes "And let the cone be cut by a plane parallel to the circle KA, and let it make as a section the circle MNX, at right angles clearly to the plane through MFN, and let the straight line MN be the common section of the circle MNX and of the triangle MFN; therefore it is the diameter of the circle. And let the straight line XL be the common section of the plane of reference and of the circle. Since then circle MNX is at right angles to triangle MFN, and the plane of reference also at right angles to triangle MFN, therefore the straight line LX, their common section, is at right angles to triangle MFN, that is to triangle KFA [the "axial triangle"]; and therefore it is perpendicular to all the straight lines touching it and in the triangle; and so it is perpendicular to both MN and AB.

> Again since a cone, whose base is the circle MNX and whose vertex is the point F, has been cut by a plane at right angles to the triangle MFN and makes as a section circle MNX, and since it has also been cut by another plane, the plane of reference, cutting the base of the cone in a straight line XL at right angles to MN which is the common section of the circle MNX and triangle MFN, and the common section of the plane of reference and of the triangle MFN, the straight line AB, is parallel to the side of the cone FKM, *therefore the resulting section of the cone in the plane of reference is a parabola* [emphasis added], and its diameter AB, and the straight lines dropped ordinatewise from the section to AB will be dropped at right angles; for they are parallel to XL which is perpendicular to AB.

To this point Apollonius has constructed a section of an equally constructed cone, and, by the way this section has been constructed, Apollonius concludes that it is a parabola. Now, and only now, *after* he has said that he has found a parabola does Apollonius go on to show that he has found the parabola having CD as its upright side (we shall not continue the proof). Apollonius's proof, one observes, is consistent with the wording of the enunciation. For there too, Apollonius *first* requires a parabola to be found and *then* specifies that it must be such that "whatever straight line is dropped from the section to the diameter at a given angle, will equal in square the rectangle contained by the straight line cut off by it from the vertex of the section and by some other given straight line." To be able to find the parabola *before* displaying the *symptoma* with its appropriate

upright side means that *what* the parabola is is independent of the *symptoma.* And from the proof of I.52, it is clear that what the parabola is is given by a careful description of a planar section of a cone—to define a parabola means to define the relative positions of a cutting plane, and the axial triangle, and the base of a cone. Put another way, constructing a parabola in a plane, the task of I.52 and I.53, comes down to finding a cone which the cutting plane cuts in the right way.[37] Needless to say, Apollonius finds a hyperbola and ellipse in propositions I.54,55 and propositions I.56,57,58 respectively, in precisely the same way.

What we have shown above is that the *symptomata* play no essential part in the *definitions* of the conic sections. But that conclusion only begs the question, what is the *function* of the *symptomata* in the *Conica*? Regarding this, one should note, first of all, that the *deductive structure* of the book does not demand that the *symptomata* appear as they do in propositions I.11–13. For the first part of I.20 and I.21 can be proven *independently* of propositions I.11–13 and can be used, in turn, to derive the properties given in I.11–13. Consider, for example, I.20, "If in a parabola two straight lines are dropped ordinatewise to the diameter, the squares on them will be to each other as the straight lines cut off by them on the diameter beginning from the vertex are to each other." We begin as in I.11 since, as we have argued, it is this initial construction which defines the parabola. So let there be a cone having vertex A and base, circle BC (see fig. 12). Let it be cut by a plane through its axis making as a section, triangle ABC, and by another plane cutting the base of the cone in a straight line DE perpendicular to BC. Let this latter plane make as a section on the conic surface the line DFE and let the diameter of the section FG be parallel to one side AC of the

[37] Referring to the same proposition, Heath admits this centrality of the cone saying, "Hence *finding* a particular conic was understood as being synonymous with *localising* it in a cone, and we actually meet with this idea in Apollonius I.52–58, where the problem of 'finding' a parabola, and ellipse, and a hyperbola satisfying certain conditions takes the form of finding a cone of which the required curves are sections. Menaechmus and his contemporaries would perhaps hardly have ventured, without such a geometrical definition, to regard the loci represented by the three equations [!] as being really curves. When however they were found to be producible by cutting a cone in a particular manner, this fact was a sort of guarantee that they were genuine curves; and there was no longer any hesitation in proceeding with the further investigation of their properties in a plane, without reference to their origin in the cone" (*Apollonius*, p. xcvii).

axial triangle. Let K and R be taken at random on the section and let KL and RS be drawn parallel to DE. We wish to show

sq.KL:sq.RS::FL:FS.

Let MN and QT be drawn parallel to BC and through the points L and S respectively. Then the plane through MN and KL and that through QT and RS are parallel to the base of the cone and, therefore, MKN and QRT are circles by proposition I.4. Accordingly, we have

sq.KL = rect.ML,LN and sq.RS = rect.QS,ST,

so that

sq.KL:sq.RS::rect.ML,LN:rect.QS,ST.

But ST = LN, therefore rect.ML,LN and rect.QS,ST are under the same height and, therefore, they are to one another in the same ratio as their bases, ML and QS. Hence,

sq.KL:sq.RS::ML:QS.

But by similar triangles ML:QS::FL:FS, so that, sq.KL:sq.RS::FL:FS, which is what we wanted to prove. Now, suppose HF be such that sq.KL = rect.HF,FL. Then since

rect.HF,FL:rect.HF,FS::FL:FS::sq.KL:sq.RS, also, sq.RS = rect.HF,FS,

and, therefore, if V is taken at random on the parabola and VU is drawn parallel to DE,

sq.VU = rect.HF,FU.

Moreover, since rect.ML,LN = sq.KL = rect.HF,FL, we have,

rect.HF,FL::rect.FA,FL::rect.ML,LN:rect.FA,FL.

But rect.ML,LN:rect.FA,FL comp.ML:FL,LN:FA[38] while by similar triangles ML:FL::BC:AC and LN:FA::BC:AB so that,

rect.ML,LN:rect.FA,FL comp.BC:AC,BC:AB.

[38] We have adopted here Taliaferro's convention of writing "rect.AB,BC:rect.CD,DE comp. AB:CD,BC,DE" for the phrase "the ratio of rectangle contained by AB and BC to the rectangle contained by CD and DE is compounded of the ratios AB:CD and of BC:DE."

Hence, we have the proportion,

HF:FA::rect.HF,FL:rect.FA,FL::sq.BC:rectAC,AB,

and this is precisely the proportion by which the upright side is defined in proposition I.11. A similar argument shows that proposition I.21 is independent of I.12 and I.13 and that the *symptomata* given in these propositions can be derived from I.21.

It is safe to assume that Apollonius knew that the *symptomata* in I.11–13 are equivalent to the properties expressed in I.20,21 since Archimedes before him and, by implication, Euclid based their work on conics on these latter properties.[39] Moreover, Apollonius employs I.20,21 in proofs of propositions far more than I.11–13 so that as *useful* properties those in I.20,21 are equally, if not more, important than the *symptomata* in I.11–13. Why, then, does Apollonius choose to begin with the properties expressed in I.11–13 and why does he refer to them as the *fundamental properties* (τὰ ἀρχικὰ συμπτώματα)? There are, we think, three answers to this question.

1) In the light of Proclus's remarks, quoted above, regarding the application of areas and the connection of that tradition with the names later geometers chose for the conic sections, it is conceivable that Apollonius was such a tradition-minded mathematician who saw virtue in associating his work with older forms. In this Heath is probably right when he writes ". . . there would be no cause for surprise if so orthodox a geometer as Apollonius intentionally harked back and sought to connect his new system of conics with the most ancient traditional methods."[40] However, while this may be true, it is certainly not a sufficient answer. For when it suits him, Apollonius shows no hesitation to use the more "modern" theory of proportion as it appears in Books V and VI of the *Elements* when he might have

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[39] See, for example, Boyer, "Analytic Geometry in the Alexandrian Age," p. 33, Heath, *History*, II,122–125, and Dijksterhuis, *Archimedes*, pp. 55ff. Taisbak has also treated the equivalence of the two formulations and the way in which the upright side enters into the picture by means of what he calls the "Theorem of conjugate application of area" (in an unpublished paper under the same title). This states "If v:h is a given ratio, and if p and d are at right angles in the ratio p:d = v:h, and if a square is applied elliptically (or hyperbolically) to d with a rectangular defect (or excess) of the form F(v:h) [a rectangle whose sides are in the ratio v:h], then the same square can be applied elliptically (or hyperbolically) to p with a defect (or excess) that is congruent and similarly situated with the first one."

[40] Heath, *Apollonius*, p. lxxxiii.

used theorems connected with the application of areas.[41] So if respect for tradition guided his approach, it did so selectively.

2) The *symptoma* refers to a single arbitrary point on a given curve by relating a single square to a single rectangle. The sides of these figures are determined by the diameter, ordinately drawn segments, and upright side, and all of these are given the moment the curve is produced in the cone. Thus, *once a conic section is given*, the *symptoma* gives an *immediate* criterion for a point to be on that conic section. This is the use of the *symptoma* one sees in propositions I.26, 32, 53. For example, in I.53 which is, as we have said, the completion of I.52, Apollonius must find a parabola whose upright side is CD as in I.52, but where the straight lines dropped ordinatewise are *not* dropped at a right angle to the diameter. Apollonius begins as follows. Let the angle HAE be made equal to the given angle and let AH = half CD (see fig. 14). Let EL be bisected by K and let KM be drawn perpendicular to EL and such that rect.LK,KM = sq.AL. Next, relying on I.52, Apollonius describes a parabola having KL as its axis and KM as its upright side and then writes ". . . [the parabola] will pass through the point A because sq.AL = rect.LK,KM," which is the *symptoma* with reference to diameter KL and parameter KM. The symptoma, of course, is not the only property that Apollonius could have used to determine whether or not a given point is on a given conic section; he could have used, for example, proposition I.42 for points on a parabola. It must be stressed, though, that in every case, *the curve will always be given beforehand* and

[41] For example, in I.15 Apollonius needs to show that if, in an ellipse, DE is drawn ordinatewise and through the center, C, of diameter AB, then DE is a conjugate diameter to AB (see fig. 13). Thus, he must show that if G is an arbitrary point on the ellipse and GW is drawn parallel to AB, then DE will bisect GW. Clearly, GX and WY being drawn ordinatewise to AB, it suffices to show XC = CY. After a rather long development, he arrives at the proportion YA:AX::XB:BY. From here, the conclusion of the proof is simple. For from the proportion YA:AX::XB:BY we have *separando* YX:AX::XY:BY so that AX = BY; but AC = CB therefore also the remainders XC = CY. However, he could have chosen to argue in a way that would only depend on "the most ancient traditional methods," in Heath's words, without significantly lengthening or complicating the proof. For from the proportion YA:AX::XB:BY we have rect.YA,YB = rect.AX,XB; then, by *Elem*.II.5, one of the key theorems for application of areas, we have rect.YA,YB together with the square on YC equals the square on AC, and rect.AX,XB together with the square on XC equals the square on CB; but the square on AC is equal to the square on CB and, from before, rect.YA,YB is equal to rect.AX,XB, therefore, the square on YC equals the square on XC and, therefore, YC = XC.

I.53 is, in this respect, no exception. This is important, for that observation forces one to reject any claim that Apollonius is using here what amounts to a *converse* of I.11, in which, like an equation, the *symptoma* would define the parabola; indeed, Apollonius does not claim that the point A lies on *a* parabola because sq.AL = rect.LK,KM but, rather, that A lies on *the* parabola *already drawn* having KL as axis and KM as upright side.

3) A related reason though, we think, a more compelling reason why Apollonius presents the *symptomata* in I.11–13 as the fundamental properties of the conic sections is that since for a given diameter, squares on the ordinates are equated to rectangles applied to a *fixed* straight line, the upright side, Apollonius can, by means of this upright side, give a kind of "signature" to a given conic section. This has some importance since, in Apollonius's way of doing things, a cone can be cut in any number of ways to produce any number of sections—and as this creates a need for a general procedure for sectioning, it also creates a need for a way of distinguishing sections and of identifying them.[42] Of course, as before, one must always keep in mind that this "signature" is given *only after the section has been determined as a parabola, hyperbola, or ellipse by the sectioning procedure, and it only reflects that procedure.* Indeed, in I.52, since the only condition on EA is that it be greater than CG (see fig. 11), the cone Apollonius constructs there is not unique. This means that the construction actually provides any number of cones in which the reference plane makes as a section a parabola with axis AB. The parabolas are all identical since the sectioning procedure *also* assures that their upright sides are always equal to CD. We first see a hint of this function of the upright side in proposition I.14 in which Apollonius defines the opposite sections. In this proposition he proves two things: 1) he shows that if a plane cuts both napes of a conic surface and does not pass through the vertex, then the sections produced are both hyperbolas and these hyperbolas have a common transverse diameter; and 2) he shows that the upright sides of the two hyperbolas are equal. The usefulness of this second fact is demonstrated almost immediately in proposition I.16. But we hardly think that it was

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[42] One recalls, as argued above, that this is *not* the same as *defining* the conic sections. The parabola, ellipse, hyperbola, or opposite sections have already been generated and identified *as* parabolas, ellipses, hyperbolas, or opposite sections by the geometric sectioning procedure.

merely as a useful lemma that Apollonius included the equality of the upright sides in the proposition *defining* the opposite sections. Rather, the equality of the upright sides points to an equality of the hyperbolas themselves.[43] And it does this through the "figure (εἶδος)" formed by the upright side and transverse diameter. For since the transverse diameter is common to both hyperbolas, the figure associated with the one hyperbola and that with the other are one and the same rectangle. Indeed, Apollonius often refers to *the* figure of the opposite sections, for example, in I.60. So, although Apollonius still refers to the opposite sections as two curves, he has now in the upright side a unifying element.

Let us summarize. Apollonius's basic understanding of the conic sections is entirely geometric; it is an understanding expressed in terms of planes and cones and their relative positions and their mutual intersections and it is *only* in these terms that Apollonius describes the generation of the conic curves. One never finds in the *Conica* the *symptomata* described in some general way independent of the cone and then used to generate a curve. In other words, Apollonius deals with *sections of cones* and not with *general curves.* Lines in Apollonius have properties. Disembodied properties in search of lines, Pirandellian creatures, do not populate Greek geometry. This makes Greek geometry, and the *Conica* in particular, nonanalytic, and this means that whatever part the *symptomata* play in the *Conica* it cannot be the part they would play if the thought behind the *Conica were* analytic.

B. *A Second Look at the Algebraic Interpretation of the Elements of Conics*

The reading of the *Conica* presented thus far differs, of course, from the usual account, the account originating in Zeuthen's *Die Lehre von den Kegelschnitten im Altertum,* which, as we noted in the last chapter, was adopted entirely by Sir Thomas Heath, and repeated without question by almost every historian of mathematics since. In Heath's words, ". . . Apollonius, in deriving the three conics from any cone cut in the most general manner, seeks to find the relation between

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[43] In Book VI, Apollonius clarifies what is meant by equal conic sections, as well as showing under what conditions given conic sections are equal. In the third chapter, we shall discuss Book VI in detail and show how the equality of conic sections is bound up with the very possibility of their being *given* in the first place.

the coordinates of any point on the curve referred to the original diameter and the tangent at its extremity as axes (in general oblique), and proceeds to deduce from this relation, when found, the other properties of the curves. *His method does not essentially differ from that of modern analytical geometry except that in Apollonius geometrical operations take the place of algebraic calculations* [emphasis added]."[44] This account of the *Conica*, then, sees the diameter and ordinate direction, given by the tangent at the vertex, as coordinate axes and the *symptomata* as algebraic equations. So, if we introduce into the Apollonian diagram for I.11, say, an oblique system of coordinates, in which the diameter of the section serves as X-axis and the tangent at the vertex of this diameter as Y-axis, while calling the *latus rectum*, the *orthia*, p, then what Apollonius proved is that $y^2 = px$ (or, if an *arbitrary* ordinate is chosen for the Y-axis, $y^2 = p(x + k)$, so that the vertex is now located at $x = -k$), and, "therefore," there is no essential difference between the analytic and the Apollonian treatment of the parabola, "trivial" differences aside namely the "cumbersome" Greek rhetoric coupled with Apollonius's "refusal" to use algebraic symbols.

At the foundation of the usual account is another conception received and baptized by Zeuthen, adopted by Heath, and repeated by almost every historian of mathematics since. We are, of course, referring to the notion of "geometrical algebra." Supporters of "geometrical algebra" see no difficulty in translating statements concerning rectangles and squares occurring in ancient texts into equations involving products and powers. Van der Waerden, for example, having defined algebra as the art of handling algebraic expressions and equations, can say that, "If this definition is applied to any Babylonian or Arab text it is unimportant what symbolism the text uses. Our relation

$$(a + b)^2 = a^2 + b^2 + 2ab$$

can be stated in words thus:

> The square of the sum is the sum of the squares of the terms and twice their product.
>
> *The statement in words says exactly the same thing as the formula* [emphasis added].[45]

[44] Heath, *Apollonius*, p. cxvi.
[45] B. L. van der Waerden, "Defence of a 'Shocking' Point of View," p. 200.

It is from this type of premise that Heath concludes that Book II of Euclid's *Elements* "... supplies the means for solving the problems of modern algebra so long as they do not involve expressions above the second degree, and provided, so far as the solution of quadratic equations is concerned, that negative and imaginary solutions are excluded ...".[46] This characterization of Book II of the *Elements* is particularly significant since this book contains the basic propositions relating to the application of areas (παρβολὴ τῶν χωρίων) which, we recall, Apollonius uses to describe the *symptomata* of the conic sections. To the extent that this characterization is plausible, then, the characterization of the *Conica* as an incipient work in analytical geometry is plausible. Since we have discussed the general issues involved in the "geometric algebra" interpretation in the last chapter, here we wish to consider the question whether the *Conica*, specifically, can be viewed as an incipient work in analytical geometry, and, in particular, whether the *symptomata* are equations and whether the diameter and ordinate directions are coordinate axes.

Symptomata *and Equations*

So, are the *symptomata* equations? To answer this, one must answer two related questions 1) are the *symptomata algebraic* in character, and 2) do the *symptomata function* as equations? As to 1), the arguments against "geometrical algebra" are, of course, as damning to an algebraic interpretation of the *Conica* as they are to an algebraic interpretation of Book II of the *Elements*. The *symptomata*, like the propositions of Euclid's second book, are stated in terms of rectangles and squares; there is no hint in *Apollonius's text* that these rectangles and squares are meant to represent the arithmetical operations of multiplication and squaring. Moreover, the fact that *one who already knows symbolic algebra* can easily rewrite the symptomata as algebraic equations does not, of course, prove that this is the way Apollonius understands the *symptomata*. Indeed, there is ample evidence *in the text* to show that when Apollonius says that a certain square equals a cer-

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[46] Heath, *Apollonius*, p. cii. See also Heath, *History*, I,153 and *Euclid's Elements*, I,346, 372–374.

tain rectangle he means just that and nothing more. Let us consider two examples.

1) In each of the "*symptomata* propositions," I.11–14, the squares on an ordinately drawn segment is compared to the rectangle applied to a certain fixed line (*lines* in the case of the opposite sections) and having as its breadth the segment cut off on the diameter by the ordinately drawn segment. In I.11, for example, the square on LK is *equal* to the rectangle contained by the fixed line HF and the line FL (see fig. 10).

Apollonius's proof, somewhat abridged, is as follows. Having drawn MN and KL (see fig. 10) parallel to BC and DE respectively in the base of the cone (where BC and DE are perpendicular by the construction of the section), Apollonius concludes (by I.4) that the plane through KL and MN, being parallel to that through BC and DE, determines a circle in the conic surface and that, therefore, rect.ML,LN = sq.KL. Apollonius has previously set up FH perpendicular to the diameter FG and has required that it be "contrived (πεποιήσθω)" that FH be such that

sq.BC:rect.BA,AC::FH:FA.

This, together with the fact, proven in *Elem.*VI.23, that

sq.BC:rect.BA,AC comp.BC:CA,BC:BA

implies that

HF:FA comp.BC:CA,BC:BA.

From here, Apollonius continues,

But

BC:CA::MN:NA::ML:LF

and

BC:BA::MN:MA::LM:MF::NL:FA.

Therefore

HF:FA comp.ML:LF,NL:FA.

But

rect.ML,LN:rect.LF,FA comp.ML:LF,LN:FA.

Therefore . . .

HF:FA::rect.ML,LN:rect.LF,FA.

But, with the straight line FL taken as common height,

HF:FA::rect.HF,FL:rect.LF,FA

therefore

rect.ML,LN:rect.LF,FA::rect.HF,FL:rect.LF,FA.

Therefore

rect.ML,LN = rect.HF,FL.

But

rect.ML,LN = sq.KL

therefore also

sq.KL = rect.HF,FL.

It is quite clear that this proof can be readily translated into algebraic language by someone who knows algebra. For such an algebra knower the correspondences between rectangles and products, ratios and fractions, compounding ratios and multiplying fractions are obvious. With KL = y, HF = p, and FL = x, the *symptoma* proven becomes the equation $y^2 = px$ where p is a *parameter*, as many authors refer to HF,[47] in the equation. Of course, it is not by accident that such authors refer to p as a *parameter*, for a "parameter" in modern usage is "a constant or variable that distinguishes special cases of a general mathematical expression,"[48] and, therefore, using the word "parameter" is natural where one sees the fundamental object in question as such a general mathematical expression or equation. *Apollonius*, however, calls HF the "straight line to which the straight lines drawn ordinatewise to the diameter are applied in square (παρ' ἣν δύνανται αἱ καταγόμεναι τεταγμένως ἐπὶ τὴν διάμετρον)," stressing, thus, the geometrical procedure in which it is involved. Indeed, Apollonius also refers to HF simply as the *orthia* (ὀρθία), the "upright side," and consistent with this name, the *orthia* is set up perpendicular to the diameter at the very outset of Apollonius's proof. Setting up the *orthia* in this way has *no algebraic interpretation and fulfills no algebraic requirement*; for an algebraic interpretation it is enough that the *orthia* be a fixed length. As the actual side of a rectangle, however, setting up the *orthia* perpendicular to the diameter from which the other side of the rectangle is cut off has obvious significance.

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[47] See, for example, Heath, *Apollonius*, pp. lxxx–lxxxi.

[48] John Daintith and R. D. Nelson, *The Penguin Dictionary of Mathematics*, p. 244.

2) Book VI of the *Conica*, one of the non-elementary books of the *Conica*, concerns the similarity and equality of conic sections. Two conic sections are said to be equal when one can be superimposed on the other in such a way that the two coincide throughout without cutting one another.[49] As we shall discuss in detail in the chapter on Book VI, the definition of equal conic sections recalls Euclid's fourth Common Notion in Book I of the *Elements*: "Things which coincide with one another are equal to one another." The notion is quite intuitive geometrically when one applies it to points and straight lines, as Euclid does in *Elem.* I.4, 8. It is somewhat more problematic when one applies it to curved lines. Thus, when Apollonius uses the definition of equal conic sections in Book VI, he reduces the procedure to that of superimposing points and rectilinear figures. In this, the importance of his expressing the *symptomata* in terms of rectangles and squares is obvious. The first two propositions in Book VI show that two parabolas are equal whenever their upright sides referred to the axis are equal and that two hyperbolas or ellipses are equal whenever the "figures" (defined in def. 10) constructed on the axes are equal. The proof of VI.1 runs as follows: Let there be two parabolas whose axes are the lines AD and FH and, first, let their upright sides AE and FM be equal (see fig. 15). Then the sections will be equal.

For suppose, when one applies the axis AD on the axis FH, some point of parabola AB does not coincide with parabola FG. Take B on the non-congruent part of AB, draw BK perpendicular to AD, and complete the rectangle KE. Further, take FL = AK, draw LG perpendicular to FL and complete the rectangle LM. Since AK and AE are equal to FL and FM, respectively, the rectangle KE must be equal to the rectangle LM. But the square on KB is equal to the rectangle KE and the square on LG is equal to the rectangle LM [here, of course, we are using the *symptoma* as it is given in I.11] from which it follows that the straight line KB is equal to the straight line LG. Hence, if we place axis AK on axis FL so that the two coincide, BK will fall on LG, and point B on point G contradicting the hypothesis that B is on the part of the parabola AB which does *not* coincide with FG. So, the parabolas must coincide throughout and are, therefore, equal. The converses are proven similarly.

[49] *Conica*, Book VI, def. 1.

One should note that the rectangles KE and LM are equal because they are superimposed on one another, AK on FL and AE on FM, so that their equality is, in fact, congruence. Apollonius demonstrates proposition VI.2 in exactly the same manner of applying one axis on another, corresponding one rectangle to one rectangle, then a square to a square, then the side of the square to the side of the square. So, not only are the *symptomata* expressed geometrically, but they are also *used* geometrically. The language of squares and rectangles which they employ leads to the comparison of *congruent* and *similar*[50] figures, that is, comparisons not only of area, but also of shape.

That the *symptomata* are not algebraic in character should now be clear. This conclusion, however, does not exclude the possibility that the *symptomata* are somehow *analogous* to equations of curves or *function* in this way. Thus, our second question above. In fact, we have already answered this question in the first part by showing that the *symptomata* alone do not define or determine the conic sections, that, in Apollonius's text, a *symptoma* never identifies an *arbitrary* curve as a parabola, hyperbola, or ellipse. Even in the (Greek) mathematics of a much later period, one does not find the *symptomata* taking on this function. In *On the Section of a Cylinder* by the fourth century A.D. mathematician, Serenus, for example, there is a series of propositions showing that a section of an oblique cylinder which is neither subcontrary nor parallel to the base has a property identical to the *symptoma* of an ellipse. While not leaving the question of its specific identity open, Serenus, significantly, includes a proposition, the very next one, showing that "it is possible to exhibit a cone and a cylinder which are alike cut in one and the same ellipse,"[51] as if to show that one can conclude only now that the section having the ellipse's *symptoma* is indeed an ellipse. Like Apollonius, then, Serenus thought that a curve having the *symptoma* of an ellipse still could not simply be called an ellipse until he could find a cone of which it was a section.

This is the place to point out the long-established difference between *symptoma* and *ousia* (οὐσία) in mathematico-philosophical discourse. A *symptoma* is a property. Properties are of something; in mathematics this is certainly the case. Mugler translates *symptoma* as: "*Proprietas, propriété, Eigenschaft, Property*" and defines it as "Fait géometrique qui

[50] In VI.12, 13.
[51] Translation from Heath, *History*, II,520.

est la conséquence de conditions réalisées par une figure."[52] He goes on to cite examples from Archimedes, Apollonius, Pappus, and Proclus, substantiating his translation. *Ousia* is, of course, something else. It is "that which is one's own, one's substance, property [!], . . . real property [!], being, existence," but "in the phil. of Plat., and still more in that of Arist., the doctrine of οὐσία still plays an important part." The meanings given in the standard *Greek Lexicon* of Liddell, Scott, and Jones, are: the being, essence, true nature of a thing, the essence, species, or true definition of a thing, reality, a primary substance, element, any organic substance, the material cause. In short, it seems that linguistic usage in the relevant mathematico-philosophical discourse lends full support to what was said above about the *symptomata versus* the definitions, essences of conic sections. The *symptomata* are not *ousiai*. The *true definitions* of the conics are provided by their generations, not by their properties.

The Locus Problem

Further evidence for what we have been saying can be found in Apollonius's treatment of the three and four-line locus problem. Apollonius mentions the locus problem in his introductory letter to Book I. There he says, concerning the contents of Book III, that it "contains many incredible theorems of use for the construction of solid loci and for limits of possibility (τοὺς διορισμούς) of which the greatest part and the most beautiful are new. And when we had grasped these, we knew that the three-line and four-line locus had not been constructed by Euclid, but only a chance part of it and that not happily. For it was not possible for this construction to be completed without the additional things found by us." The theorems in Book III which relate most closely to the three and four-line locus are propositions III.54–56, the last three propositions of the book. Let us consider III.54. It states: "If two tangents to a section of a cone or to a circumference of a circle meet, and through the points of contact parallels to the tangents are drawn, and from the points of contact, to the same point of the line of the section, straight lines are drawn across cutting the parallels, then the rectangle contained by the straight lines cut off to the square on the straight line joining

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[52] Mugler, *Dictionaire Historique*, p. 393.

the points of contact has a ratio compounded of the ratio which the inside segment line joining the point of meeting of the tangents and the midpoint of the straight line joining the points of contact has in square to the remainder, and of the ratio which the rectangle contained by the tangents has to the fourth part of the square on the straight line joining the points of contact." Thus, in the diagram (see fig. 16), if ABC is the section, AD and CD are tangents meeting at point D, E is the mid-point of AC, AF and CG are parallel to CD and AD respectively, H is an arbitrary point on the section, and CH and AH are extended to F and G respectively, then the proposition states that the ratio rect.AF,CG:sq.AC is compounded of the ratios sq.EB:sq.BD and rect.AD,DC:fourth sq.AC.

As Taliaferro rightly points out,[53] the three-line locus property for all but the opposite sections follows easily from this proposition. For if HY, HX, and HZ are drawn from H to AC parallel to AD, BE, and DC respectively, then we have CZ:ZH::AC:AF and AY:YH::AC:CG by similar triangles and from these proportions and *Elem.*VI.23 we have

rect.CZ,AY:rect.ZH,HY::sq.AC:rect.AF,CG.

But from *Conica* III.54, sq.AC:rect.AF,CG is a fixed ratio, being compounded of fixed ratios and, therefore the ratio rect.CZ,AY:rect.ZH,NY is also fixed. By a similar argument we obtain the proportion rect ZH,HY:sq.HX::rect.CD,DA:sq.DE so that rect.ZH,HY:sq.HX, too, is a fixed ratio. Therefore, the ratio rect.CZ,AY:sq.HX is, *ex aequali*, a fixed ratio. Now since HY is parallel to AD, AY can be thought of as the distance from H to line AD and, similarly, CZ can be thought of as the distance from H to CD. Hence, if H is an arbitrary point of section ABC, the rectangle whose sides are the distances from H to the lines AD and CD bears a constant ratio to the square whose side is the distance from H to the line AC. Moreover, if the distance from H to, say, AD is taken at a different angle, say, along HR, then the theorem still holds. For if one draws line PY parallel to HR then, noting that PY:AY is a fixed ratio, again we have rect.CZ,PY:sq.HX is constant though it is not necessarily the same ratio as rect.CZ,AY:sq.HX.

Assuming that this reconstruction is correct, what does it show?

[53] R. Catesby Taliaferro, *On Conic Sections*, p. 799.

To answer this, consider what Proclus has to say about loci and locus-problems: "I call 'locus-theorems' those in which the same property (σύμπτωμα) occurs throughout the whole of a certain locus, and I call 'locus' a position of a line or a surface producing (ποιοῦσαν) one and the same property. Some locus-theorems refer to lines, others to surfaces; and since some lines are plane and others solid—plane lines being those which, like the straight line, lie in a plane and whose generation is simple, and solid lines those which are produced by some sectioning of a solid figure, like the cylindrical helix and the conic lines—I should say further that of locus-theorems referring to lines some have a plane and others a solid locus."[54] Proclus's description of loci and locus-theorems makes a clear distinction between the property possessed by the curve and the curve itself and, therefore, by implication, its geometric origin. The property, therefore, cannot be understood as *generating* the curve or in any way defining the curve. Thus, he says that the position of the line *produces* the property, and *not* the other way around.[55] Proclus's discussion of loci is prompted by what he says is the first locus theorem in the *Elements*, namely, *Elem.*I.35. This is the proposition stating that "Parallelograms which are on the same base and in the same parallels are equal to one another." Did Euclid see I.35 as *defining* parallel lines? Obviously not! In the same way, an ellipse, parabola, and hyperbola are *defined* as sections of a cone, although at the same time they answer the question, what curve has the property that if from any of the points on it distances are taken at fixed angles to fixed lines, the rectangle whose sides are two of the distances bears a fixed ratio to the square whose side is the third distance. Hence, when in

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[54] Friedlein, pp. 394–395, Morrow's translation, pp. 310–311.

[55] Ivor Thomas translates ποιοῦσαν simply as "has," "a position of a line or surface which *has* [emphasis added] throughout one and the same property" (*Greek Mathematical Works*, I,491). One is used to thinking of the locus as the set of points satisfying or conforming to a certain condition. In this sense, the condition or property determines the points on the curve. This is particularly true in analytical geometry: the set of all points whose distance from a given point is fixed, is the set of all pairs (x,y) satisfying the equation $(x - a)^2 + (y - b)^2 = r^2$, and this equation, which only expresses the given condition, *is* the circle. Yet, this way of looking at loci is obviously precisely the opposite of Proclus's way. Our guess is that this is why Thomas chose the rather neutral translation of "has" for ποιοῦσαν; by leaving the translation ambiguous we are left to decide whether the line "has" the property because the property can be proven to belong to the line or because the property itself produced the line; Thomas, it seems, prefers the latter alternative.

his prefatory letter Apollonius speaks of constructing the three and four-line locus he is not referring to an alternate way of producing conic sections, but to the demonstration that the points on a given conic section provide a solution to the locus problem; the conic section itself must be *produced* according to its *geometric* definition.

The way in which Descartes approached Pappus's challenge of generalizing the three and four-line locus, what has come to be known as the "Pappus problem," played, to be sure, a pivotal role in the development of analytical geometry—and, indeed, one sees in it the immense gap between the explicitly algebraic thinking of *La Géométrie* and Apollonius's and Pappus's geometrical thinking.[56] Not only does Descartes, by means of his "arithmetic of lines,"[57] transform Apollonius's and Pappus's rectangles and squares into "line products," but he also changes the very conception of the problem. For Apollonius, the problem is to find if a certain locus is provided by a given curve, and although a point of the locus may first be found through an "analysis," the real solution of the problem is in the "synthesis" where the curve is constructed geometrically. For Descartes, however, the problem is *to find a point* satisfying the given conditions and, as he says, "since there is always an infinite number of different points satisfying these requirements, it is also required to discover and trace the curve containing all such points."[58] The shift from identifying a curve to finding a point may appear insignificant, especially to modern readers of Descartes, but in fact it is crucial. To understand why, one must ask what it means *to find a point* or, rather, how one is to answer the question, *where* is a point. For Apollonius, answering this means establishing that the point is "given in position (τῇ θέσει δεδόσθαι)"[59] and this is possible when the point is on a curve given in position, which implies that the curve

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[56] Significant support for the views presented here on Apollonius's and Pappus's approach to the locus problem and about the difference between this and Descartes' approach can be found in David R. Lachterman's *The Ethics of Geometry: A Genealogy of Modernity*, esp., chapter 3, section III, pp. 141–187.

[57] In the facsimile edition translated by David Eugene Smith and Marcia L. Latham, *The Geometry of René Descartes*, pp. 2–17.

[58] *Ibid.*, p. 22.

[59] About this, Taisbak has said "Points are thought of as separate entities, each with its own identity; each point is different from all other points. Now, the only question one can meaningfully ask of a point's identity is: Where is it? To describe this individuality, Euclid chose the phrase 'given in position' . . ." ("Elements of Euclid's *Data*," p. 142).

has been *constructed* along the lines, say, of *Conica* I.52–60. In Descartes' approach to the problem, a point is found, as expressed by the opening sentence of his book, through "a knowledge of the lengths of certain straight lines . . .,"[60] in this case, the lines drawn from the point to the given lines (the three and four lines in the three and four-line locus or the n lines in the generalized problem). Therefore, finding a point is independent of the curve on which it lies. It is only because there is an infinity of points satisfying the locus condition that one thinks to look for a curve. The points on a curve are, thus, *prior* to the curve itself; where a point is is governed solely by the "arithmetical" relationship among lengths drawn from the point to given lines and this same relationship, therefore, determines the curve on which the point lies. While Descartes is not entirely explicit about this, he is certainly aware that this understanding of the origin of curves is an unavoidable outcome of his new algebraic approach to the Pappus problem and that this approach embodies a radically new understanding of the problem. Indeed, immediately after the initial discussion of the problem he launches into a book-long discussion of the nature of curves.

Intentiones Primae *and* Intentiones Secundae

The view that the origins of analytical geometry lie in the *Conica* arises, to a great degree, through an underestimation of the conceptual differences between the algebraic approach to problems such as the three and four-line locus, adopted by mathematicians like Descartes, and the purely geometrical approach of the Greeks. In his various writings, Jacob Klein makes this difference particularly clear by drawing on the distinction made in medieval logic between *intentiones primae* and *intentiones secundae.*[61] *Intentiones primae* are concepts that refer to any thing other than a concept as such. "Red," "family," "circle," are all *intentiones primae*, for all refer to distinct entities. *Intentiones secundae*, on the other hand, are concepts that refer to other concepts. Thus, "quality," "relationship," "shape," are *intentiones secundae*

[60] *The Geometry*, p. 2.

[61] Jacob Klein, "The World of Physics and the 'Natural' World," pp. 28–30, "Modern Rationalism," pp. 60–63, *Greek Mathematical Thought and the Origin of Algebra*, pp. 174–175, 192, 206ff., 299–300. For a general discussion of *intentio prima* and *intentio secunda* see Norman Kretzmann, "History of Semantics," pp. 369f.

for we say "red is a quality" and "a circle is a shape." According to Klein, ancient mathematics is dominated by *intentiones primae.* A Greek mathematician always looks at a circle, a parabola, or a spiral, and when, for example, he classifies curves, he classifies *distinct* curves before his eyes; he does not attempt to classify all the curves that *can be.* This dominance of *intentiones primae* is also the reason, evidently, why Greek geometry is a visual science that cannot do without diagrams: a proof in Greek geometry focuses on some distinct object "imaged," as Klein says, in the diagram.[62] So, characterizing Greek mathematics as a mathematics dealing with *intentiones primae* means that its geometrical approach does not arise out of the abstract problem of incommensurable magnitudes, say, but rather *out of its direct confrontation with the objects of its study.* Modern mathematics, on the other hand, is characterized by a kind of synthesis of *intentiones primae* and *intentiones secundae.* Modern mathematics becomes modern when it becomes symbolic and it becomes symbolic only when *intentiones secundae* become *intentiones primae.*[63] Here is how Klein puts it: ". . . if we have an expression such as a + 2ab + c, what does 'a' mean? 'a' *can* be 4, or 6, or 150, or any possible one, but 'a' *is not* 4, or 6, or 150, or any other one. Thus, 'a' doesn't mean certain objects, namely units, the multitude of which it indicates, but rather the *concept* of the number as a multitude of units. So, at first glance, 'a' belongs to the concepts of the second class, to the class of concepts which are applied not to individual objects but to concepts themselves. This, however, is only the first step. Actually, we deal with 'a' and with all such algebraic numbers in exactly the same way as arithmetic deals with ordinary numbers. In other words, in algebra we use concepts of the second class as though they were concepts of the first class. We identify, in the process of algebraic thinking, concepts of the first class with concepts of the second class. What we call a symbol is nothing else but a concept of the second class interpreted as a concept of the first class. Therefore, we can now state that the 'peculiar kind of abstraction' used in algebra is a *symbolic* abstraction."[64] In this light, Descartes' *concepts* of "abscissa" and "ordinate" bear the same relation to Apollonius's "correspond-

[62] Cf. Klein, "World of Physics," pp. 27–28.
[63] *Ibid.*
[64] Klein, "Modern Rationalism," pp. 62–63.

ing" *objects* as "a" bears to "4," or "6," or "150." For they do not stand for some particular abscissa and ordinate in some distinct ellipse or hyperbola; the x and y axes are not lines such as FL and LK in fig. 10, but the very *concept* of such lines, the "general procedure"[65] by which *any* such FL and LK are produced. Yet these axes are treated as if they were distinct particular objects and so is any object built with their assistance. For this reason Descartes (and *we*) can speak of "$y^2 = px$" as the object called a parabola, since in the sense elucidated above "$y^2 = px$" *is* a parabola. In other words, the algebraic representation of a curve has become the curve itself.[66]

The Place of Diameters

The difference between Apollonius's purely geometrical approach to the study of conic sections and an algebraic approach such as that of Descartes can be seen readily in Apollonius's treatment of the diameters of conic sections. As we have seen, Apollonius defines the diameter of a curve in general at the start of the *Conica.* In the porism to proposition I.7, we learn that a section of a cone, in particular, has a diameter. The exact enunciation of I.7 is as follows: "If a cone is cut by a plane through the axis, and if it is also cut by another plane cutting the plane the base of the cone is in, in a straight line perpendicular either to the base of the axial triangle or to it produced, then the straight lines drawn from the resulting section on the cone's surface, made by the cutting plane, parallel to the straight line perpendicular to the base of the triangle will fall on the common section of the cutting plane and of the axial triangle, and further produced to the other side of the section, are bisected by the common section; and if it is a right cone the straight line in the base will be perpendicular to the common section of the cutting plane and of the axial triangle, and if oblique, it will not always be perpendicular, but whenever the plane through the axis is perpendicular to the base of the cone." One observes that the word "diameter" appears nowhere in this enunciation nor, for that matter, in the proof which follows it. Only after the proof, almost incidentally, does Apollonius say in the form of a porism that ". . . from

[65] Klein, "World of Physics," pp. 28–29.
[66] Cf. Klein, *Greek Mathematical Thought,* p. 123.

this [proposition] it is evident that the straight line FG [which is the common section of the cutting plane and the axial triangle] is the diameter of the section DFE, since it bisects the straight lines drawn parallel to some straight line DE [the common section of the cutting plane and the plane of the cone's base], and that it is possible for some parallels to be bisected by the diameter FG and not be perpendicular." Hence, the focus of proposition I.7 is a *sectioning procedure* which happens to contain *within it* the construction of a diameter and the determination of the corresponding ordinate direction. Accordingly, when Apollonius applies I.7 in I.11–14 he does so with the same detailed description of how the various planes must be oriented with respect to the cone, and it is through this description that we know how to cut the cone to produce the desired section. The reader of these propositions sees, therefore, that *the production of the conic sections is simultaneous with the production of a diameter.* In this sense, the diameter is *inseparable* from the conic section in which it is found, and conversely, the definition of a conic section, as the section of a cone, includes the position of its diameter.

At first sight, this might appear at odds with what is certainly one of the main themes of Book I, namely, that every conic has an unlimited number of diameters. The way Apollonius develops this theme, however, only *emphasizes* the distinction of the diameter produced by the sectioning procedure of I.7. In the first step of this development, Apollonius proves the existence of a conjugate (συζυγεῖς) diameter for the ellipse (I.15) and opposite sections (I.16). It is important to note that beginning with the conjugate diameter is not required by the book's logical structure: proposition I.47, which finally shows that an ellipse has an unlimited number of diameters, does not depend on proposition I.15. Far from it, one can easily imagine a sequence in which I.15 follows I.47; for since the one case I.47 does not cover is that of the conjugate diameter, I.15 would, in such a sequence, be the completion of I.47. However, Apollonius does not proceed in this way because he wants to maintain, from the start, as close a connection as possible between the original diameter and any new diameter. Hence, he begins with the *conjugate* diameter, being as the name suggests, a new diameter *wedded* (σύζυγος) to the original diameter.

The next step in Apollonius's development of this theme is the set of propositions, I.46–48. These show, in effect, that any line parallel to the diameter of a parabola is itself a diameter and any line

(with certain qualification required by I.43 on which I.47 relies) from a point on an ellipse, hyperbola, or opposite sections through the center is a diameter. Significantly, Apollonius does not call these new diameters "diameters" in propositions I.46–48; nor does he call them "diameters" in I.49–51, which further develop I.46–48. Let us look at I.47: "If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and through the point of contact and the center a straight line is drawn in the direction of the section, it bisects the straight lines drawn in the section parallel to the tangent." Here is Apollonius's proof: "Let there be an hyperbola or ellipse or circumference of a circle whose diameter is the straight line AB and center C (see fig. 17), and let the straight line CE be joined and produced, and let a point N be taken at random on the section, and through N let the straight line HNOG be drawn parallel [to DE].

I say that NO = OG.

For let the straight lines XNF, BL and GMK be dropped ordinatewise. Therefore by the things already shown in the forty-third theorem,

trgl.HNF = quadr.LBFX,

and,

trgl.GHK = quadr.LBKM.

Therefore the remainders quadr.NGKF = quadr.MKFX; Let the common pentagon ONFKM be subtracted; therefore the remainders trgl.OMG = trgl.NXO. And the straight line MG is parallel to the straight line NX; therefore NO = OG.

The proof of the proposition rests on I.43, and its diagrams are almost identical to those accompanying I.43. This earlier proposition is not at all easy to understand especially with its long enunciation. However, the picture becomes clearer when one realizes that most of the enunciation of I.43 is, in fact, a description of the diagram, and the diagram is full of symmetries: two lines from the center to the curve, two tangents, two lines through a point on the curve parallel to the two tangents. I.47 extends the symmetry of the diagram in I.43 by showing that CE is a "diameter" as CB, that is, CE bisects all chords parallel to the tangent at B. Apollonius develops this symmetry in I.49–51. In these propositions, he shows that *symptomata* like those described in terms of the original diameters can be described in terms of the new diameters. At the end of I.51, Apollonius finally makes all this explicit, calling the lines in I.46–51

"diameters" for the first time and confirming the symmetry of the roles: "And with these things shown, it is at once evident that in the parabola each of the straight lines drawn off parallel to the original diameter is a diameter, but in the hyperbola and ellipse and opposite sections each of the straight lines drawn through the center is a diameter; and that in the parabola the straight lines dropped to each of the diameters parallel to the tangents will equal in square the rectangles applied to it, but in the hyperbola and opposite section they will equal in square the areas applied to it and exceeding by the same figure, but in the ellipse the areas applied to it and defective by the same figure; *and that all the things which have been already proved about the sections as following when the principal diameters* (τῶν ἀρχικῶν διαμέτρων) *are used, will also, those very same things, follow when the other diameters are taken* [emphasis added]." What is important to see here is that although these new diameters share the same general properties as the original diameter they do not yet share the same status. Apollonius still refers to the original diameter, the diameter produced together with the sectioning procedure, as the *principal* diameter, and, thus, separates it from the others. In this sense, one cannot really speak of an *arbitrary* diameter in the context of the *Conica*. Indeed, in the proofs of I.46–51 and their lemmas I.42–43, the principal diameter is given in position with the section, and the new diameter can be produced only if the tangent at its vertex meets the original diameter. And conversely, when in I.52–60 Apollonius wants to produce a section having some line as its diameter, he produces a section in a cone and shows that either the given line is the principal diameter itself or it is related to the principal diameter in the way prescribed in I.46–51. In Book II, one might argue, Apollonius speaks about *a* diameter, and, therefore, an *arbitrary* diameter, for example, in II.44 where Apollonius poses the problem "Given a section of a cone, to find a diameter." However, II.44 is not a proposition in isolation; Apollonius is free to speak of *a* diameter at this point only because he has developed the relationship of *any* diameter to the *principal* diameter in Book I.

So, put in precise terms, what Apollonius does in Book I amounts to showing that in every conic there exists an indefinite number of diameters *in addition* to the principal diameters, that is, in addition to the diameters that come together with the genesis of the curves in the cone. The distinction between the additional diameters and the principal diameters has no importance in the proof of theorems,

as Apollonius himself implies at the end of his summary of I.46–51. Yet, immediately following this summary are the propositions in which Apollonius shows how conic sections may be produced having diameters in various positions. And, of course, in every case the diameter on which the constructions depend are *principal* diameters. Towards the end of Book II, Apollonius shows us, more or less, the converse of these propositions. Thus, recalling that the angle between the tangent and the diameter gives the ordinate direction, we have II.51 stating "Given a section of a cone, to draw a tangent which with the diameter drawn through the point of contact will contain an angle equal to a given acute angle." II.51 is really enough for all applications, for it shows how in any given section one can find the diameter one needs to prove some fact or another. What the propositions at the end of Book I, and with them the special status of the principal diameter, show, we think, is Apollonius's concern for the place of the diameter in the production of the curves in the cone. And is this not a purely *geometric* concern? For such a concern arises only in one who conceives of the conics as things which cannot be detached or abstracted from their geometric origin. *The* diameter is first of all the *principal* diameter for Apollonius, since this diameter is one which can be seen directly in the geometrical description of the curves. Otherwise, it is hard to understand why after so carefully developing the symmetry of the roles played by all the diameters, Apollonius should still talk about a principal diameter and show such diameters produced in the cone.

Conclusion

To understand the *Conica* historically presupposes understanding its discourse as a geometrical discourse in which showing, pointing, and drawing play crucial roles. This is especially true regarding the propositions which, for Apollonius, are the *elements* of conics. The text makes this manifest in three ways. *First*, the terms of the text are either themselves simple geometric entities, points, lines, planes, circles, rectangles, and so on, or they are based on such entities—and this may be by means of analogy, as the diameter, center, and radius of a conic are analogous to diameter, center, and radius of a circle. These terms are, of course *the* basic elements of the text; they do not represent some other more basic elements. Thus the *symptomata*,

for example, distinguish one conic from another by comparing elementary squares and rectangles *as such*. *Second*, the approach is entirely visual in spirit. This is a mathematics which cannot be abstracted from the diagrams which show its content.[67] One can hardly find a proposition which is not accompanied by a diagram, and when one does find a proposition without a diagram, it invariably refers to the diagram of *another* proposition. The diagrams and the text constitute an inviolable dialectical unity guiding, constraining, and shaping one another. Apollonius prefers speaking about the cone rather than the conic surface, the opposite sections as two curves instead of one, where these preferences are determined largely by the degree to which such things are portrayable by a diagram. *Third*, there is a concern, not always immediate but always present, with the sensible production of a curve and its parts. This concern means that each curve *as a whole* is seen as the result of a concatenation of simple, visually appealing, relationships: intersections of planes with planes, intersections and arrangements of lines. Thus, the special status of the principal diameter comes through its close connection with its visible genesis. Together, these three aspects show not only that Apollonius's elements of conic sections are geometrical in nature but also in what sense they are. This makes the text an essentially different kind of text from an algebraic text such as *La Géométrie*, regarding which Newton wrote in a marginal remark: "Error, error non est Geometria!"

Before we turn to the *non-elementary* books of the *Conica*, we need to examine the *fourth* of Apollonius's books of elements of conic sections, and to show that its character *as a book of elements* is consistent with what we set out thus far. So, let us now take a look at Apollonius's book about how conic sections meet in the plane, Book IV of the *Conica*.

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[67] Cf. W. Knorr's discussion of Theaetetus 147d in *The Evolution of the Euclidean Elements*, pp. 69–74.

Figures

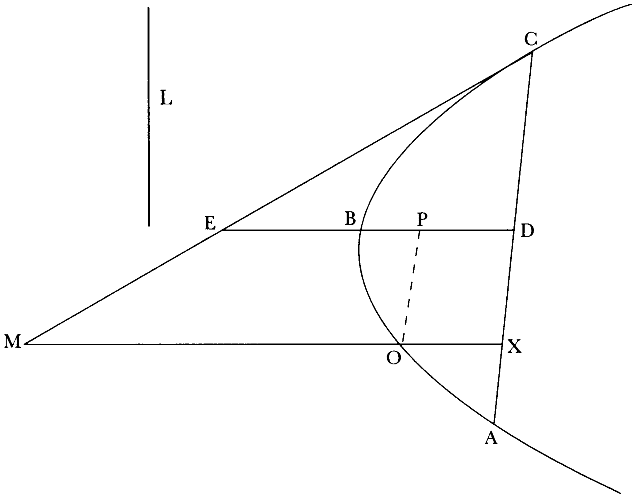

Fig. 1

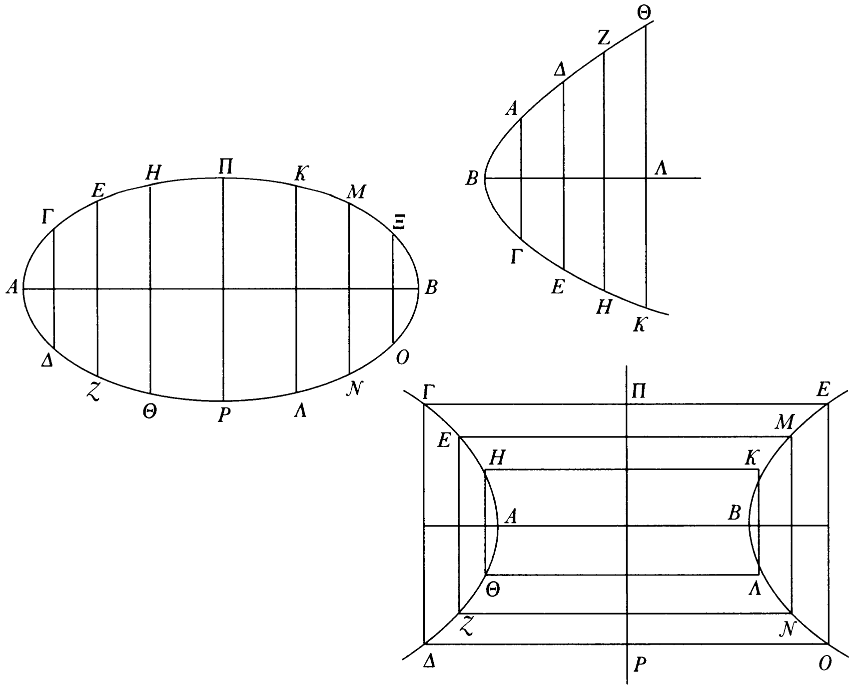

Fig. 2

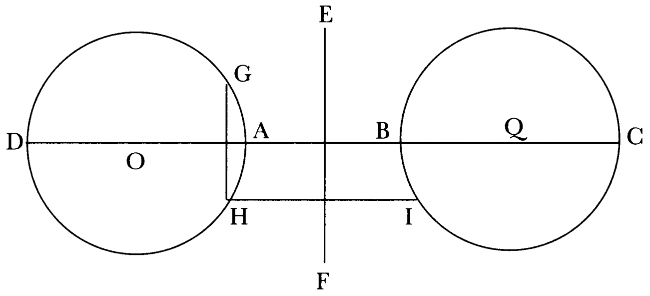

Fig. 3

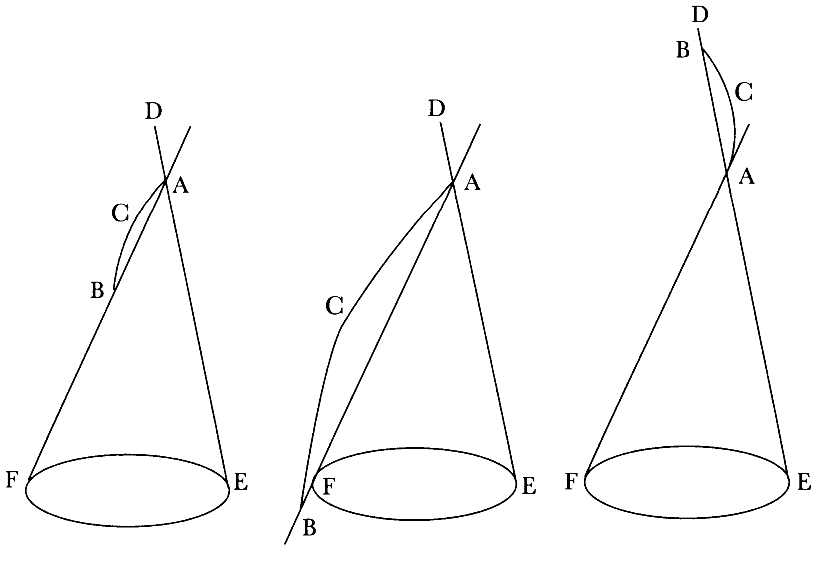

Fig. 4

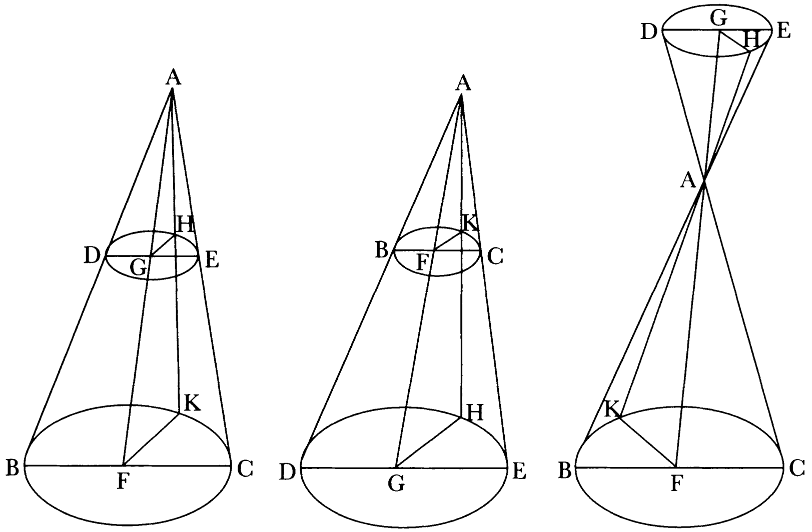

Fig. 5

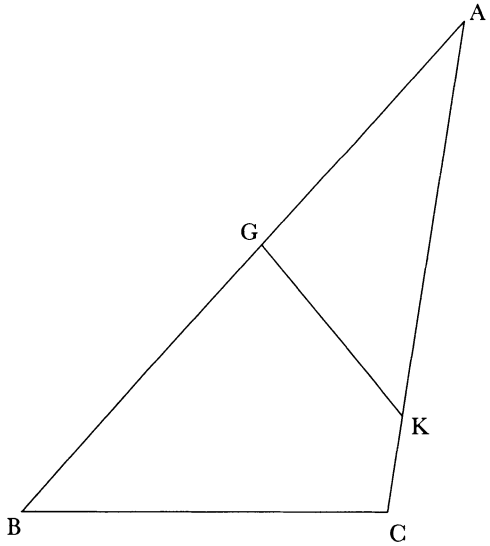

Fig. 6

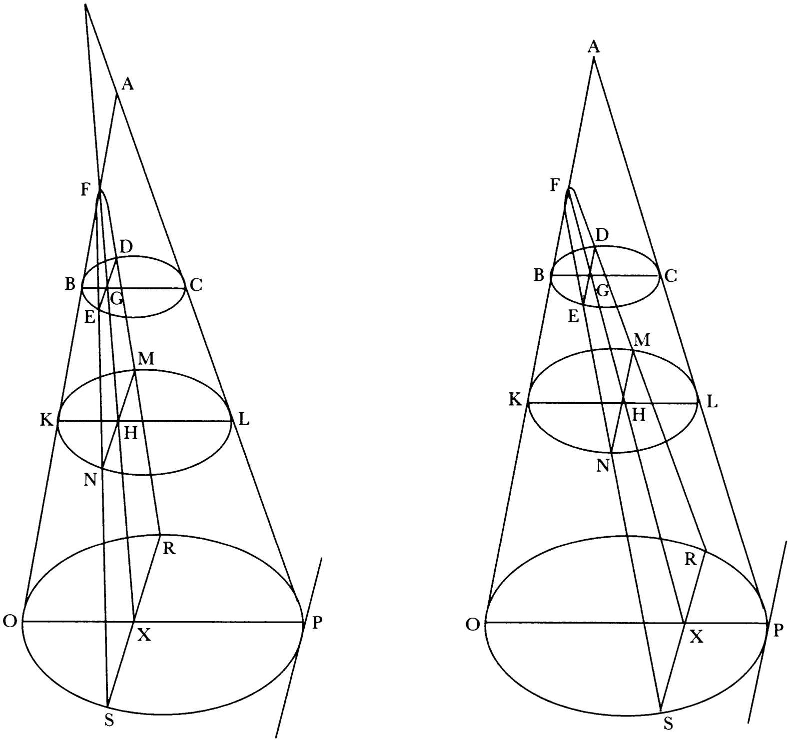

Fig. 7

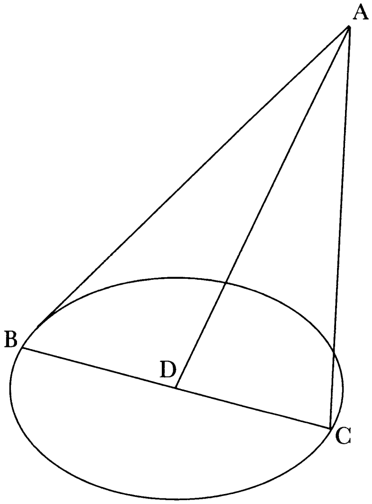

Fig. 8

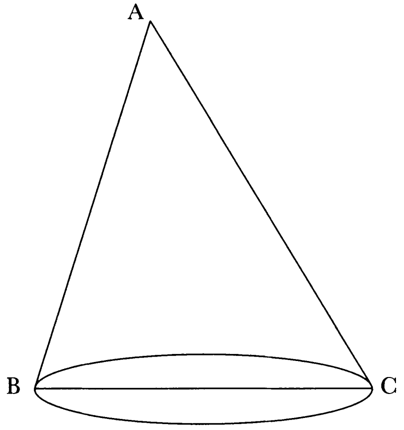

Fig. 9

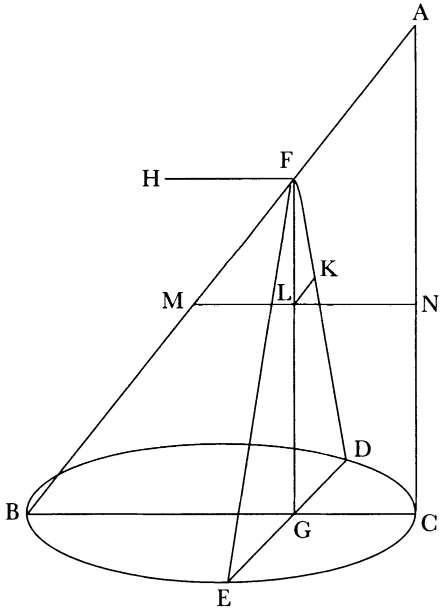

Fig. 10

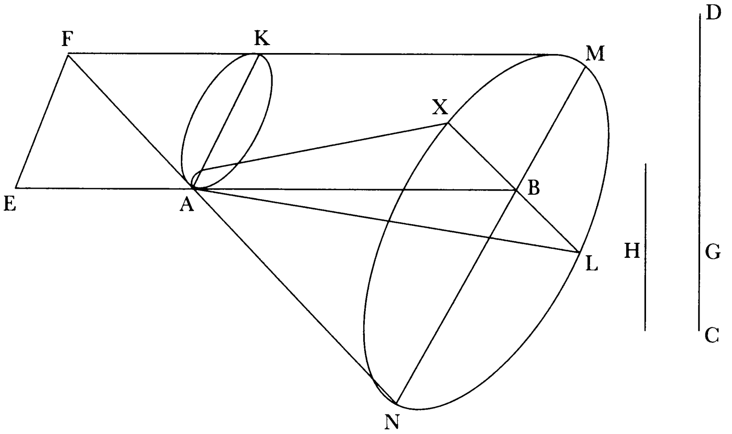

Fig. 11

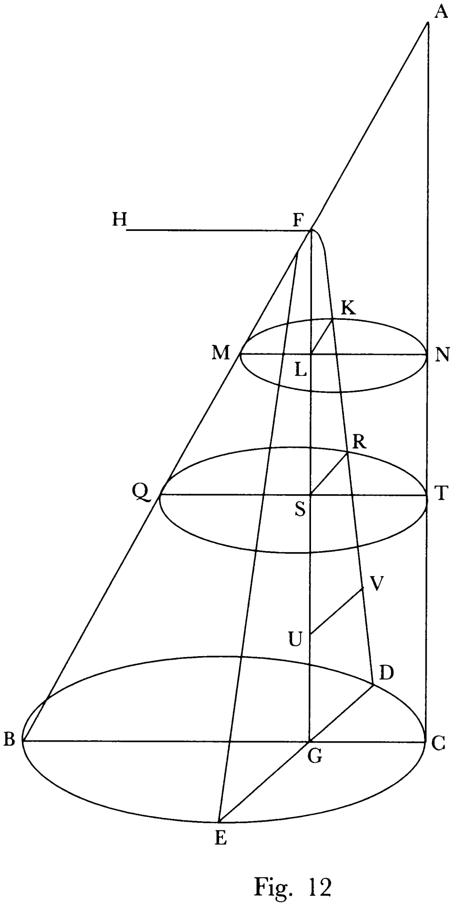

Fig. 12

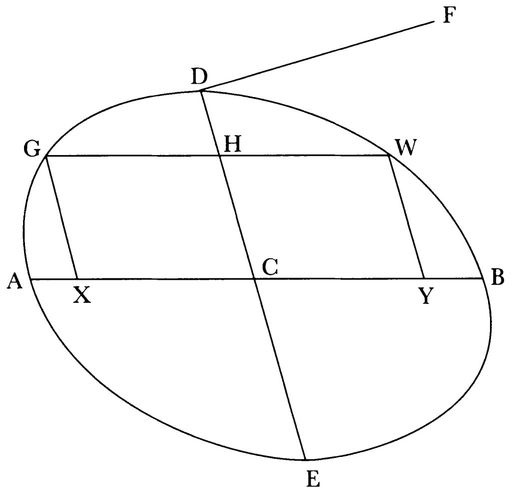

Fig. 13

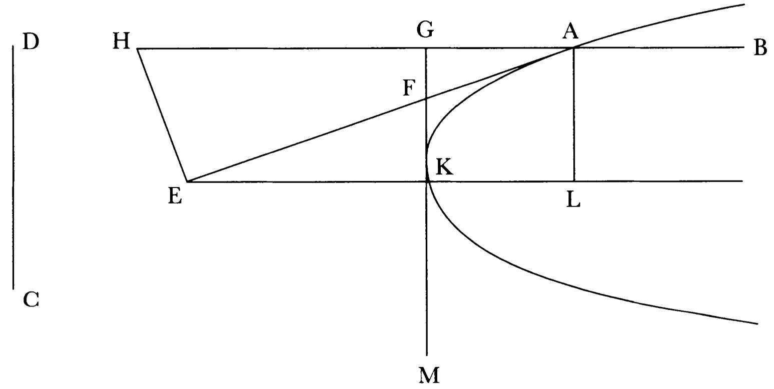

Fig. 14

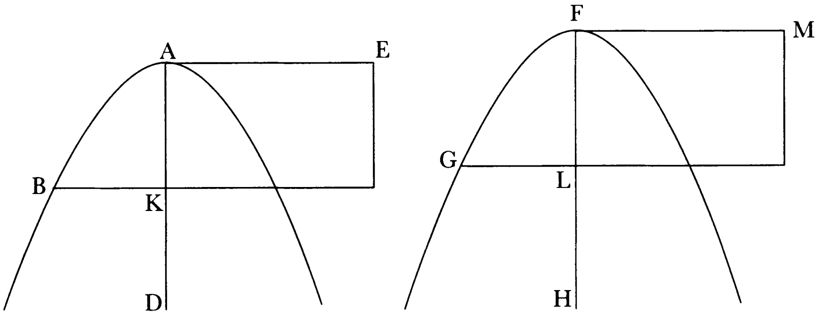

Fig. 15

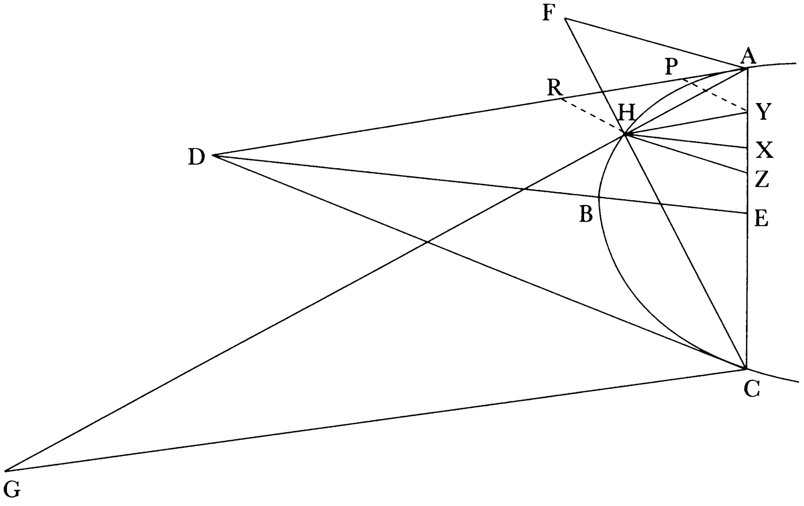
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Fig. 16

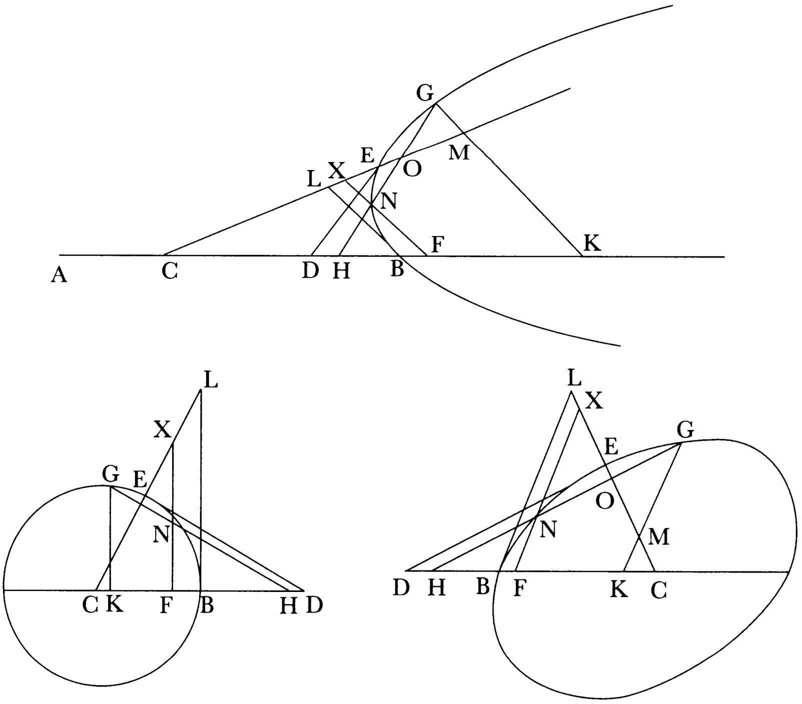
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Fig. 17

CHAPTER THREE

BOOK IV OF THE ELEMENTS OF CONIC SECTIONS: HOW CONIC SECTIONS MEET IN THE PLANE

Introduction

Relative to the other books in the *Conica*, Book IV has been neglected by modern readers of Apollonius. Taliaferro, for example, chose to translate only Books I to III into English.[1] No one seems to have an authoritative answer as to why Taliaferro left Book IV untranslated, but it is hard to avoid the conclusion that he saw it as a project having no great urgency. Heath does summarize the contents of Book IV in his *History of Greek Mathematics*, but he also leaves the reader with the clear message that there is no pressing need to study the book, for, as he says, "Book IV is on the whole dull, and need not be noticed at length [!]"[2] Audacious, if not reckless, remarks such as that of Heath give us justification enough to dedicate a separate chapter to Book IV, but we must also do so in order to complete the work begun in the last chapter. In that chapter, we chose to examine the character of Apollonius's "course in the elements" only by looking at Books I to III since those books by themselves could sufficiently bring to light the fundamentally geometric outlook of the elements. But Apollonius's "course in the elements" also includes Book IV. So, with the conclusions of the last chapter in hand, we must now ask What is the place of Book IV among the elements? This is the guiding question of the present chapter, and we believe that, in asking it, the reader of Apollonius's treatise best approaches the content and structure of the book. Indeed, this must be true if what we said in the last chapter is true, namely, that the "course in the elements" is the framework or setting for Apollonius's thought about conics.

On the face of it, the elementary character of Book IV is not

[1] We have included an English translation of Book IV in the appendix to this book.

[2] *History of Greek Mathematics*, II, 157.

entirely clear. For one, the propositions of Book IV are never used again, at least not in the *extant* books of the *Conica.* Perhaps, Apollonius used results from Book IV in Book VIII, the lost book of determinate problems, for he does say in the prefatory letter to Book IV that "The investigation of these theorems [in Book IV] is . . . of considerable use in the synthesis of problems and limits of possibility." Even allowing this speculation, though, it is unlikely that any of these results could have had an *essential* place in Book VIII, especially since the book that Apollonius mentioned as important for Book VIII was not Book IV but *Book VII.*[3] In any case, no possible use of propositions from Book IV in that final book could compare to the extensive and crucial application of propositions from Books I to III throughout the *Conica.* In this way, Book IV is more like Books V to VII than it is like Books I to III. It is more like Books V to VII also in the singleness of its theme and its tight thematic structure. Indeed, one could easily ask why Book IV completes the "course in the elements" rather than, say, Book VI? After all, Book VI concerns the equality and similarity of conic sections, and, therefore, concerns a subject that, superficially,[4] appears no less fundamental than the subject of Book IV, namely, the meeting of conic sections; certainly, equality and similarity have an important place in *Euclid's Elements.* And since, like all the later books, Book VI depends only on propositions from Books I to III, it and Book IV could be switched without violating the deductive structure of the entire work.

So, why does Apollonius, nevertheless, include Book IV among the elements? What is it about Book IV that makes it "pertinent to the understanding of the whole of the science,"[5] to recall Proclus's criterion for a book of elements? We suggest three possible answers to this question. The first has to do with the analogy between propositions in Book IV and propositions in Book III of the *Elements,* the second, with the development of ideas connected specifically with

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[3] Halley, in his reconstruction of Book VIII, applies no proposition from Book IV. This, of course, does not prove that propositions from Book IV were absent also in *Apollonius's* Book VIII, only that in a reasonable reconstruction of the last book, one does *not* necessarily *expect* propositions from the fourth book, the way one *does* from the seventh.

[4] In our chapter on Book VI we shall discuss the other side of this question, namely, why Book VI is a *non*-elementary book. The question will turn on the conceptual difficulty of similarity when the objects concerned are, like conic sections, not simple.

[5] Proclus, *In Eucl.*, p. 72.

the opposite sections, and the third, with the fundamental issue of how conic sections can be together in one plane. The three possibilities are not exclusive; they may all be true to different degrees, and together they may present a better argument for the elementary character of Book IV than any one of them taken alone.

In the present chapter, we shall consider only the last two of these possibilities. As for the Euclidean analogy, we have already seen how connections with objects and ideas from Euclid's *Elements* play an important part in the elementary character of Books I to III, and in the chapters to come we shall see that they continue to play an important part in giving shape to the *Conica.* For this reason, we postpone our discussion of Book III of the *Elements* and Book IV of the *Conics* until later when we treat the analogy between circles and conic sections in the *Conica* in depth. We also postpone this discussion because the very fact that such an analogy exists *throughout* Apollonius's work means that the analogy between propositions in Euclid's book on circles and propositions in Book IV cannot explain *decisively* why Book IV should be grouped with Books I to III rather than with Books V to VIII; it is *a* reason but it cannot be *the* reason. So, we shall begin the chapter by showing how the opposite sections are a major concern in Book IV. Obviously, this is a discussion that will reinforce, and partly overlap with what has already been said about the opposite sections in the last chapter. Following this, we shall turn to the question of how conic sections can meet, or even just *be* together, in one plane. This question clearly underlies the explicit goal of Book IV, namely, to show in how many ways the sections of a cone can intersect or touch one another or the circumference of a circle. This part will also include a short digression on how Descartes treats the meeting of curves as manifest in his method for finding the normal to a curve.

The Investigation of Opposite Sections

In the last chapter, we stressed that Apollonius understands curves in a strictly geometric framework, so that the plural-singular nature of the opposite sections makes them an object of particular fascination, but also the source of a certain uneasiness, which one senses already in the first book of the *Conica.* In the fourth book, Apollonius appears surer that the opposite sections are, in fact, two curves.

Indeed, there are many places in Book IV where the opposite sections are referred to as two curves *explicitly*. In proposition IV.15, for instance, he begins, "If in opposite sections a point is taken *between the two sections* (μεταξὺ τῶν δύο τομῶν) [emphasis added] . . ." This does not mean that all the previous uneasiness is settled and gone. It may mean, on the contrary, that, in this book, Apollonius is being especially careful in treating the opposite sections and in separating them out as a special case needing special attention.

This can be seen clearly in the letter to Attalus, which opens Book IV. There, Apollonius reports that a certain Nicoteles (who, Knorr suggests, might be, in fact, none other than Nicomedes)[6] criticized Conon's treatment of the problem of the greatest number of points a conic section or circumference of a circle can meet another conic section. In the course of his criticism of Conon, Nicoteles is said to have remarked that the case in which a conic section meets opposite sections could be solved, but, as Apollonius makes sure to say, neither Nicoteles nor anyone else provided a demonstration. As for the greatest number of points in which opposite sections can meet opposite sections, Apollonius says that no one has ever noticed this question, let alone treated it.[7] So, in this, his quasi-historical review,[8] Apollonius makes it plain that this is a book that will make the investigation of the opposite sections a priority. And that promise is fulfilled not only in the sheer number of propositions dedicated to the inves-

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[6] Knorr, *The Ancient Tradition of Geometric Problems*, p. 282, note 41.

[7] The exchange is interesting since it shows that the existence of the opposite sections was known quite early. This can be inferred from the reference to Conon of Samos; we know that Conon died before Archimedes published his *Quadrature of a Parabola*, as Archimedes himself tells us in the letter introducing that work. This means, assuming that Nicoteles and Conon were contemporaries, that the *existence* of the opposite sections was known by the middle of the third century, that is, around the time of Apollonius's birth. And that suggests that the *double conic surface* was also understood to some extent at this time. The point is important not only because it suggests a possible non-Apollonian provenance for the Apollonian cone, but also because it gives some evidence for the traditional chronology regarding Euclid, Archimedes, and Apollonius; for Euclid speaks only of a cone as being produced by a right triangle rotated around one of its legs, so that, at least in the *Elements*, he shows no familiarity with the double cone. thus, it must be inferred that Euclid was writing sometime earlier than Conon, and, accordingly, earlier than Archimedes.

[8] According to Mansfeld, such historical reviews were, in the systematics of later antiquity, considered appropriate in that part of proems which discuss the *theme* of the work at hand. In this way, Apollonius's prefatory letters, he says, anticipate this later standard practice (see *Prolegomena Mathematica*, pp. 4, 38).

tigation of opposite sections, but also in the very architecture of the book, which seems to serve the purpose of emphasizing the opposite sections.

Consider the opening propositions of the book. Proposition IV.1 states: "If a point be taken outside a section of a cone or circumference of a circle, and from this same [point] two lines are extended towards the section, of which one touches the [the section] and the other cuts [the section] at two points, and [if] the line cut off inside [the curve] be divided in that ratio which the whole [line] cut off has to the part outside bounded between the point and the curve, so that the homologous lines are at the same point, [then] the line drawn from the point of contact to the [point of] division [i.e. the point dividing the inside segment] will intersect the curve, and the line drawn from the point of intersection to the outside point will touch the curve."[9] Thus, if D is the point outside of the conic section ABG (see fig. 1), and DB touches the section at B, while DEG cuts the section at two points E and G, and if DEG is divided at a point Z such that GZ:ZE::GD:DE, and the line BZ is joined and extended, then BZ will cut the section again at a point A, while the line DA drawn from A to D will touch the section at A. On the face of it this is simply the converse of III.37, which states: "If two straight lines touching a section of a cone or circumference of a circle or opposite sections meet, and a straight line is joined to their points of contact, and from the point of meeting of the tangents some straight line is drawn across cutting the line (of the section) at two points, then as the whole straight line is to the straight line cut off outside, so will the segments produced by the straight line joining the points of contact be to each other." Accordingly, the proof of IV.1 is a proof by contradiction in which the key fact is III.37. Here is how IV.1 proceeds: "For let DA be drawn from D touching the section, and let BA be joined cutting EG, if possible, not at Z, but at H. Now, since BD,DA touch [the section], BA [is drawn through] the points of contact, and GD goes through [AB], cutting the section at G,E and [intersecting] AB at H, as GD is to DE, so is GH to HE.[10] But this is absurd, for it was assumed that as GD

[9] All quotations from Book IV are from Fried's English translation included in this volume.

[10] By III.37.

is to DE so is GZ to ZE. Therefore, BA does not cut GE in a different point [from Z]; therefore, [it cuts GE] at Z."

Since proposition III.37 refers to "a conic section or a circumference of a circle *or opposite sections* [emphasis added]," one would expect "opposite sections" to appear also in the converse. But IV.1 refers only to "a conic section" or a circle. And in the next proposition, Apollonius continues, almost teasingly, by saying "The things above were demonstrated *for all the sections together* [emphasis added]," even though nothing so far has been said about the *opposite sections*. Apollonius could well have mentioned the opposite sections in the enunciation to IV.1, while furnishing the proofs related to them later, as he often does,[11] and as he does in fact here. Instead, the opposite sections enter quietly in IV.4, and the cases concerning them unfold little by little.

Three suggestions can be made as to why Apollonius proceeds this way. First, having mentioned the opposite sections explicitly in III.37, he might want to draw attention to the opposite sections here precisely by having the reader note their *absence* in IV.1. Second, by first saying the demonstrations are valid "for all the sections together," perhaps Apollonius was subtly mocking predecessors such as Nicoteles, who, because they thought they had a correct demonstration for the greatest number of points in which conic sections could meet, thought they had a demonstration for the case of the opposite sections as well, as is implied in Apollonius's prefatory letter to the book.[12] Indeed, in that same letter, Apollonius emphasizes that one like Nicoteles *could not* have had a correct demonstration for the case of opposite sections, for he says, "And all these things just spoken of

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[11] For example, and this is a random example, I.54 and I.55: I.54 requires that an hyperbola be produced in a given plane with a given diameter and a given ordinate direction, but demonstrates only the case where the given diameter is an axis, that is, where the angle between the diameter and an ordinate is right; I.55 then takes up the case where the angle between the diameter and ordinate is not right.

[12] That Apollonius should mock predecessors is not at all unlikely. Knorr points out that "Apollonius is notably ungenerous in his comments on the achievements of such precursors as Euclid, Conon, and Nicoteles. As for contemporaries, he omits referring to any of them by name, save for his Pergamese friends" (*Ancient Tradition of Geometric Problems*, p. 329). From this, among other things, Knorr concludes that ". . . the mathematical field at the close of the 3rd century takes on the appearance of an active and contentious environment. One might look on the needless personal frictions with disapproval and regret, and doubtless many of the ancient participants did so too. Yet the outgrowth of all this competition was a phenomenal advancement of geometry" (*ibid.*).

[namely, the demonstration of the greatest number of points conic sections or a circumference of a circle can meet opposite sections and of the greatest number of points opposite sections can meet one another], whose [demonstrations] I have not found anywhere, require many and various striking theorems, of which most happen to be presented in the first three books [of my treatise on conics], and the rest in this book." The third suggestion, and we think the most compelling one, is that Apollonius set off the case of the initial conic sections from that of the opposite sections in order to compare the former with the latter, to assimilate the peculiar opposite sections to the more familiar and easier to comprehend parabola, ellipse, and hyperbola.[13] So, let us see how Apollonius progresses from conic sections to opposite sections in these opening propositions of Book IV.

Proposition IV.2 should hardly be called a proposition at all. It has no diagram and its only purpose seems to be to *explain* that IV.1 is valid for the hyperbola, so long as the point D is inside the angle contained by the asymptotes. Nevertheless, that the hyperbola is now seen *within its asymptotes* is a significant step in moving from the hyperbola to the opposite sections. For although when the asymptotes are introduced in II.1 it is long after the opposite sections themselves are defined in I.14, the asymptotes are, nevertheless, essential for properly comprehending the opposite sections. First of all, together with the transverse diameter, the asymptotes unify the opposite sections, since, as we learn in II.15, the asymptotes of opposite sections are common.[14] The asymptotes also allow one to find one's way

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[13] While it is still unclear whether the opposite sections should be taken as a single entity or as two hyperbolas, it is not to be doubted that what Apollonius calls "hyperbola" is the *single* curve produced in I.12. For Apollonius, a hyperbola *has* an opposite section, but it is not the two opposite sections *together*, as it is for us. This is an error that Heath makes in translating IV.42, for example: Apollonius writes "If a hyperbola intersects each [of two] opposite sections, *its opposite section* [emphasis added] will intersect neither of the opposite sections at two points," whereas Heath, good modern that he is, writes "If one branch of a hyperbola meet both branches of another hyperbola, the opposite branch of the former will not meet either branch of the second in two points." By calling a hyperbola a curve with two branches, Heath does away with Apollonius's distinction between the "hyperbola" introduced in I.12 and the "opposite sections" introduced in I.14. Needless to say, looking at the hyperbola as one curve with two branches is natural in analytic geometry, which guides Heath's view of conics, for, in analytic geometry, what a hyperbola is is the set of *all* points in the Cartesian plane satisfying the equation (in one form or another) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

[14] The asymptotes, in this sense, also unify the opposite conjugate sections. Indeed,

around the opposite sections: they are crucial lines of reference in the geography of the opposite sections. In proposition II.33 and IV.42, for example, Apollonius speaks quite literally of "three places" (τῶν τριῶν τόπων) defined by means of the asymptotes, one place being that contained by the angle containing the hyperbola, and the two others being those contained by the angles adjacent to the angle containing the hyperbola. In the propositions opening Book IV, however, the places defined by the asymptotes are the place contained by the angle containing the hyperbola, the places contained by the angle adjacent to the angle containing the hyperbola, and the place on the asymptote itself. So, proposition IV.2, in giving point D with a definite position with respect to the asymptotes, gives it a place with respect to the opposite sections as well, and allows Apollonius to move naturally from IV.2 to proposition IV.4 where D is placed *in the angle adjacent to that contained by the asymptotes* (see fig. 2) and then to IV.5 where D is placed *on the asymptote* (see fig. 3).

Passing thus from IV.2 to IV.4,[15] one is drawn from the hyperbola to the opposite sections, for in IV.4, with D in its new position (see fig. 2), BZ, joining the point of contact B with the point Z dividing the segment EG such that GD:DE::GZ:ZE, does not intersect the initial hyperbola again, but *its opposite section.* It is surprising, in this light, that IV.5 comes when it does rather than between IV.2 and IV.4, that is, in such a way that D would pass from being *within* the asymptotes (*i.e.* within the angle containing the hyperbola), to being *on* the asymptote itself, and finally to being *outside* the asymptotes (*i.e.* within the angle *adjacent* to that containing the hyperbola). It must be emphasized, in this connection, that Apollonius is truly free to choose the order of the propositions here, that is, he is not constrained by any deductive necessity; there is no *logical* dependence among IV.2,4,5. One reason for Apollonius's placing IV.5 after IV.4 might be that proposition IV.5, in which BZ is shown to remain

in II.17, which is the first proposition concerning the conjugate sections after their definition in I.60, Apollonius proves that "The asymptotes of conjugate opposite sections are common." However, unlike the opposite sections, the opposite sections together with their conjugate sections are never considered in any way as a single entity. This, no doubt, is because *the pair* cannot be produced as a section of a double conical surface.

[15] We do not mention proposition IV.3 since it only adds a second case to IV.2, namely, where the points E and G (see fig. 4) do not contain the point of contact B. It adds little to what has already been shown in IV.1 and IV.2; it may well be a later interpolation.

parallel to the asymptote (see fig. 3), is not a converse of III.37, but of III.35.[16] Being thus related to a *different* proposition in Book III, Apollonius would have reason to separate IV.5 from IV.2,4. The problem with this is that IV.5 is *not* separated from IV.2,4. Proposition IV.5 has no new *protasis*, and, hence, it must be viewed as belonging to *a single series* beginning with IV.1. This means that, although not mentioned explicitly in the proposition, the opposite section must be thought to appear nevertheless, for it has been introduced in IV.4, a fact strengthened by the words "The same things being the case . . .," with which proposition IV.5 begins.[17] When Apollonius says in IV.5 that BZ will be parallel to the asymptote, therefore, one must understand, in particular, that BZ has been shown *not to cut the opposite section.* Indeed, with the opposite section of the hyperbola in place, there are three possibilities for a line to cut these opposite sections: it can cut one of the opposite sections in two points, cut each of the opposite sections in one point, or cut one of the opposite sections in one point. Proposition IV.4 may be viewed, therefore, as a positive analogy to IV.2, with BZ cutting the hyperbola in two points altogether in the latter and the opposite section in two points altogether in the former. Proposition IV.5, on the other hand, may be viewed as the *negative* analogy of these, a contrast to them, being that BZ cuts *neither* the hyperbola again *nor* its opposite section. In its role of a negative analogy, thus, it is rightly placed farther from IV.2 than is IV.4.

Starting with IV.1, then, which is stated only for the sections of a cone (*i.e.* for the parabola, hyperbola, and ellipse) and the circumference of a circle, Apollonius brings us to the rich possibilities of the opposite sections by taking us through a series of analogies, and analogies to the analogies. Thus, proposition IV.6, which states, "If a point be taken outside a hyperbola, and from this [point] two lines be drawn to the section, one of which touches [the section] while the other is parallel to one of the asymptotes, and [if the part

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[16] Proposition III.35 tells us that if a point is taken on one of the asymptotes and a straight line from it touches the section, and through the point of contact a parallel to the asymptote is drawn, then "if from the point taken some straight line is drawn across cutting the section at two points, then as the whole straight line is to the straight line cut off outside, so will the segments of the straight line cut off inside [by the line drawn through the point of contact parallel to the asymptote] be to each other."

[17] Appropriately, Heiberg's diagram for IV.5 shows the opposite section.

of the latter line] inside the section be set equal to the [part] cut off between the section and the point, [then] the line joined from the point of contact [of the former line] to the produced point will intersect the section, and [the line] drawn from the point of intersection to the point outside will touch the section," begins a sequence (IV.6,7,8) that follows the same pattern as that in IV.2,4,5, even employing identically labeled diagrams. Proposition IV.9 brings us back to IV.1 in a more obvious way. It states: "If from the same point two lines be drawn each cutting a section of a cone or circumference of a circle in two points, and [if] the segments cut off inside be divided [in the same ratio] as the wholes are to the segments cut off outside, so that the homologous lines are at the same point, [then] the line drawn through the points of division will intersect the section in two points, and [lines] drawn from the points of intersection to the point outside will touch the curve." Thus, whereas in IV.1 we have a line touching a conic section and a line cutting the section in two points, in IV.9 we have, from a point D outside the section, two lines DQE and DHZ (see fig. 5) *each* cutting the section at two points, Q,E and H,Z, respectively; the proposition then says that if DQE and DHZ are divided at points L and K such that EL:LQ::DE:QD and ZK:KH::DZ:DH, then, the line joining L,K will cut the section at two points B,A and that the lines DB and DA will touch the section. Like proposition IV.1, IV.9 depends on III.37, and, like IV.1, it too fails to mention the opposite sections in its enunciation. In fact, the pattern of propositions following IV.9 is so much like that following IV.1 that it is hard to imagine any other explanation than that Apollonius fully intended the reader to go mentally once more through the whole geometrically genetic passage from the parabola, hyperbola, ellipse, and circle to the various cases involving the opposite sections and take note again of the various analogies among them.[18]

[18] Seeing IV.9 in this light shows the inadequacy of Heath's view of the purpose of IV.9, and the propositions following it, namely, that they provide a method of drawing two tangents to a conic section from a given point outside the section (*History of Greek Mathematics*, II,157–8). One might arrive at such an interpretation of IV.9 if, ignoring its place in the general flow of propositions opening Book IV, and, of course, its role in the proof of IV.25, first main result of Book IV, one were to isolate IV.9 from the rest of Book IV. But even so, the interpretation would remain problematic. First, Apollonius mentions no such method in the letter introducing Book IV or in the letter introducing Book I. Second, and certainly more

The progression from the more familiar conic sections, the parabola, ellipse, and hyperbola, through the hyperbola taken separately, and, finally, to the opposite sections marks not only the sequences of propositions up until IV.25, but also the overall structure of the book, which is punctuated by the propositions IV.25,38, and 55. Accordingly, IV.25 states that "A section of a cone does not cut a section of a cone or circumference of a circle at more points than four," then IV.38 states that "A section of a cone or circumference of a circle will not intersect opposite sections at more points than four," and, completing the sequence, IV.55 states that "Opposite sections will not intersect opposite sections at more points than four." Of course, here, unlike the first part of the book, there is a certain measure of logical dependence. In particular, proposition IV.55, as one might expect, depends heavily on previous propositions, namely, propositions IV.37,38,41,42,43,44,45, and 46. However, these all belong to the part of the book concerning how conic sections meet the opposite sections, and all those after IV.40 until IV.55 concern, specifically, the ways a *hyperbola* meets opposite sections and how the opposite section of the hyperbola meets opposite sections.[19] It ought to be observed, also, that Book IV's deductive structure is such that dependence on previous propositions is nowhere greater than in propositions IV.55,56,57,[20] giving further weight to the part of the book concerning the opposite sections and adding momentum towards it.

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damning, Apollonius has *already* shown in II.49 how to draw a tangent to a conic section from a point outside of the section, and he emphasizes there (in the case of the hyperbola at least) that, by the same method, a *second* tangent could also be drawn to the section.

[19] The presence of IV.37 here may cause one to suspect that the group of propositions between IV.38 and IV.55 follow the group beginning with IV.25 for logical rather than thematic reasons. But proposition IV.37, which states that "If a section of a cone or circumference of a circle intersects one of the opposite sections, it will not intersect the remaining section at more points than two," as well as IV.36, which states that "If a section of a cone or circumference of a circle intersects one of the opposite sections at two points, and the curves between the points of intersection have their concavity in the same direction, then the curve, being produced at the points of intersection, will not intersect the other opposite section," are clearly propositions that are related to IV.38. Proposition IV.37 does depend on a previous proposition, proposition IV.35; IV.35, however, depends on no other proposition in Book IV, so one might also include it among the propositions leading up to IV.38 rather than among those following IV.25.

[20] All of these propositions come from those following IV.36, and only two of the latter depend on prior propositions (IV.44 depends on IV.9 and IV.48 depends on IV.1).

Both in the small and in the large, then, Apollonius's arrow is aimed towards the opposite sections in Book IV. Why this gives the book its elementary character has in a sense already been answered by the previous chapter. For the opposite sections are inextricably linked with the general double-conical surface, which, despite Apollonius's uneasiness with it, still remains the most elemental of all of Apollonius's elements of conic sections. The mere existence of the opposite sections is a constant reminder of that ultimate source of all the conic sections, for, unlike the other sections, the opposite sections cannot be conceived without the double-conical surface. And it is in a book of elements, as we argued in the last chapter, that one's attention should be turned towards the geometric source of the conic sections. It is not surprising, therefore, that the prominence of the opposite sections throughout Books I to IV is characteristic of those books alone; in Books V to VII, by contrast, the opposite sections *appear explicitly only twice*, once in Book VI and once in Book VII.[21]

But, of course, Book IV is not *about* opposite sections. It is about how conic sections meet in a plane. Does this theme itself have an elementary character? We think it does. For it too forces us to reflect on the conic sections' geometric origins. So, let us then turn now to the guiding question of Book IV, "in how many ways the sections of a cone meet[22] each other and the circumference of a circle . . ."

How Conic Sections Meet in a Plane

Although *the* question of Book IV is what is the greatest number of points at which conic sections and opposite sections can meet one another or a circumference of a circle, Apollonius is well aware that to answer that he must consider *how* conic sections meet: whether

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[21] Proposition VI.16 states that "Opposite sections are similar and equal," and VII.31 states that "When a pair of conjugate diameters is drawn in an ellipse or between conjugate opposite sections, then the parallelogram bounded by that pair of diameters with angles equal to the angles formed by the diameter at the center is equal to the rectangle bounded by the two axes." Of course, the opposite sections appear *implicitly* each time Apollonius speaks of the transverse diameter of a hyperbola.

[22] We are changing Taliaferro's translation here from the general introductory letter to Book I. Taliaferro has "intersect" where we have "meet," but Apollonius uses the verb συμβάλλω not τέμνω, and, of course, the book considers more than how conics sections intersect or cut one another.

they touch or whether they cut; if they touch at how many points do they cut; if they meet at one place, can they meet at another. We can get a feeling for the variety of configurations Apollonius takes into account by going through a proposition such as IV.56.[23]

Proposition IV.56 states that "If opposite sections touch (ἐπιψαύωσιν) opposite section at one point, they will not intersect at more than two other points." Here is Apollonius's demonstration:

> Let there be opposite sections AB,BG and others, D,EZ,[24] let BGD touch (ἐφαπτέσθω) AB at B, let them have their convexity turned oppositely, and, first, let BGD intersect GD at two points G,D, as in the first figure (σχήματος) (see fig. 6).
>
> Indeed, since BGD cuts [GD] at two [points] containing convexities turned oppositely, EZ will not intersect AB. Again, since BGD touches AB at B, having their convexity turned oppositely, EZ will not intersect GD. Therefore, EZ will not intersect either of the sections AB,GD; therefore, [the sections] will meet at the two [points] G,D, only.
>
> But let BG cut GD at one point G, as in the second figure (fig. 7). Therefore, EZ will not intersect GD, whereas it will intersect AB at one point only. For if EZ meets AB at two [points], BG will not intersect GD. But it was assumed that they meet at one [point].
>
> If BG does not intersect section D, as in the third figure (fig. 8), then, by what has been said [above], EZ will not intersect D, whereas EZ will not intersect AB at more points than two.
>
> If the sections have their concavities in the same [direction] (figs. 9, 10, 11), the same demonstrations will apply.
>
> So, from the demonstrations, what was proposed is clear in all possible configurations (διαστολάς).

There are two things in this and the neighboring propositions especially worth observing. The first is the presence of the word διαστολή and the second is the various terms Apollonius employs for the meeting of conic sections. The word διαστολή, which, in this context, means ordering, arrangement, or configuration, is rare in the mathematical literature.[25] Indeed, besides propositions IV.55,56, and 57, Mugler cites *no other* place where it appears.[26] Its use in IV.55–57, therefore, underlines what we remarked at the outset, namely, that,

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[23] For details on the steps of the proof, see the notes to the translation in the Appendix.

[24] That is, AB and D is one pair of opposite sections and BG and EZ is a second pair. See the note on this point in the translation of Book IV in the Appendix.

[25] See Mugler, *Dictionnaire Historique*, pp. 137–138. Mugler gives τάξις as a synonym.

[26] *Ibid.*

in Book IV, Apollonius is interested not only in the narrow question of what are the number of points at which conic sections can meet but also the more general question of how they may be together in one plane, *i.e.* what is their planar *configuration.* This is also entirely consistent with another slight deviation from standard practice seen in these propositions, namely, Apollonius's referring *explicitly* to figures.[27] Although in Greek geometry the diagram or figure is always the main focus of the demonstration, usually what is referred to is what is *in* the diagram, that is, its component parts, as such, as an object of reference. Here, however, Apollonius refers directly to the first or second or third *figure* (σχῆμα).[28] This has the effect of turning our attention not to this or that pair of opposite sections, but to *the two together*, how they lie with respect to one another in the single drawing. And clearly the ways the opposite sections lie with respect to one another are precisely those configurations indicated by the word διαστολή.

A further sign of the attention Apollonius gives to configuration is his speaking about the convex (τὰ κυρτά) or concave parts (τὰ κοῖλα) being in the same direction or turned oppositely. Unlike διαστολή, of course, "the convex" and "the concave" are not unusual in Greek mathematical works,[29] but in the *Conica* they appear *only* in Book IV, and quite frequently there. Apollonius's first use of the phrase "having its convexity in the same direction" is in IV.35, where he must prove that "A section of a cone or circumference of a circle will not intersect a section of a cone or circumference of a circle, not having its convexity in the same direction, at more than two points." Apollonius's argument is telling. Suppose a section of a cone or circumference of a circle ABG (see fig. 12) meets a section of a cone or circumference of a circle ADBEG at more than two points, say, A, B, and G, then, Apollonius writes, "Since three points A, B,

[27] For a Euclidean example, see *Elem.* III.33.

[28] In IV.55, as in *Elem.* III.33, it is quite literally the *drawing* (καταγραφή).

[29] The words "concave" and "convex" appear, for example, in Euclid, *Elem.* III.8, and the phrase "having concave parts in the same direction" is defined by Archimedes both for a curved line and a surface in his *On the Sphere and the Cylinder* (definitions 2 and 4). For a curved line, Archimedes' definition is this: "I call such a [curved] line [that lies entirely on one side of the straight line joining its extremities, that is, a line such as that described in Archimedes' def.1] concave in the same direction if, two points whatsoever being taken, all the straight lines between the points fall on the same side of the line, [where] some [may be] on one and the same side or along the [line] itself, but none on the other side."

G have been taken on the curve ABG, if AB,BG are joined, they will contain an angle having its concavity in the same direction as that of the curve ABG. For the same reason ABG contains an angle whose concavity is in the same direction as that of curve ADBEG. Therefore, the curves we have been speaking of have both their concave and convex parts in the same direction, which is impossible." One might say there is an implicit dependence on a definition of "having concavity in the same direction" such as that found in Archimedes' *Sphere and Cylinder*,[30] but, more directly, Apollonius seems just to assume that any angle in such a curve "having its concavity in the same direction" will have *its* concavity in the same direction, that is, it will *open in the same direction.* The contradiction is then that an angle will open in *two different directions.* Apollonius uses these phrases connected with concavity and convexity as characteristics not of the curved lines themselves but of their relative positions in the plane; they are phrases very much connected, therefore, with configurations precisely such as those distinguished in IV.56.

Our second observation regarding IV.56, though it applies no less to Book IV as a whole, has to do with the words Apollonius uses to speak about curves meeting. The point simply is this, that he uses *so many* words. Consider the first case in IV.56, in which BG touches AB at one point A and cuts D at two points G,D (Fig. 6):

> Indeed, since BGD cuts (τέμνει) [GD] at two [points] containing convexities turned oppositely, EZ will not intersect (συμπεσεῖται) AB. Again, since BGD touches (ἐφάπτεται) AB at B, having their convexity turned oppositely, EZ will not intersect (συμπεσεῖται) GD. Therefore, EZ will not intersect (συμπεσεῖται) either of the sections AB,GD; therefore, [the sections] will meet (συμβάλλουσιν) at the two [points] G,D, only.

In this passage and elsewhere in the book, we find three different ways of speaking about curves meeting: for curves that *cut* or *intersect* one another, Apollonius uses τέμνειν or συμπίπτειν; for curves that *touch* one another, he uses ἐπάπτεσθαι or ἐπιψαύειν; and for curves that *meet* in some unspecified way, συμβάλλειν or, sometimes, again, συμπίπτειν. Apollonius is not altogether precise in his employment of these words. For example, the first clause of the quotation above, ". . . since BGD cuts (τέμνει) [GD] at two [points] containing convexities turned oppositely, EZ will not intersect (συμπεσεῖται)

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[30] See the previous note.

AB . . .," is plainly referring to proposition IV.41, which states, "If a hyperbola intersects (συμπίπτῃ) one of the opposite sections in two points having its convexity turned oppositely, [then] the opposite section [of the hyperbola] will not intersect (συμπεσεῖται) the other opposite section." One might explain the change from "intersects (συμπίπτῃ)" in IV.41 to "cuts (τεμνει)" in IV.56 by saying that IV.41 refers *generally* to "meeting," which, of course, covers the *special* case of meeting in the first part of IV.56, namely, "cutting." But this argument does not work since Apollonius introduces that first case in IV.56 by saying, ". . . first, let BGD *intersect* (συμπιπτέτω) [emphasis added] GD at two points G, D, as in the first figure"!

Such imprecision and superfluity in Apollonius's use of words for meeting might be thought to flaw his text. But that is so only when one assumes these words to be *strict* mathematical terms. Like the word "tangent" or "asymptote,"[31] none of the words used by Apollonius in Book IV for meeting, cutting, or touching (*i.e. tangent*) has any other definition than the implicit "everyday" one, certainly none has a *stated* definition anywhere in the *Conica.* They seem, rather, to be used descriptively, used, that is, to give general descriptions of the configurations drawn in figures such as those in IV.56. This should be said somewhat more strongly: *cutting* and *touching* are, for Apollonius, exactly the kind of thing one *sees* in a diagram, they are seen as *basic and qualitatively different situations of curves.* Precision, of course, arises in the expansion of cases presented in the *proven* propositions. To make this clear, one ought to contrast this geometric diagram-centered view with the algebraic perspective of Descartes. So, let us digress for a moment and look at Descartes' method for "drawing straight lines falling at right angles [on curves] at whatever points on them one chooses."[32]

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[31] We return to this issue of the lack of definition for "tangent" and "asymptote" in our chapter on conics and circles.

[32] Descartes, *La Géométrie*, pp. 341ff. The translation here is ours; all subsequent translations are by David E. Smith and Marcia L. Latham in their *Geometry of René Descartes.* Page numbers, however, continue to be according to the original, which is reprinted in Smith and Latham's edition.

Digression: Descartes' Treatment of Cutting and Touching in La Géometrie

To start, Descartes considers an arbitrary curve, CE, and a point on it, C. The problem, then, is to draw through C a line normal to the curve CE (see fig. 13). In classic analytic style, he says: "Suppose the problem solved, and let the required line be CP. Produce CP to meet the straight line GA, to whose points the points of CE are to be related."[33] With CM drawn perpendicular to GA, we let $MA = y$ and $CM = x$. We must now find an "equation expressing the relation between x and y." This equation, of course, is *not* the equation of the curve but an equation relating x, y, *and* the position of P which determines the position of the normal.[34] Accordingly, Descartes continues and says, ". . . let $PC = s$, $PA = v$, whence $PM = v - y$. Since PMC is a right triangle, we see that s^2, the square of the hypotenuse, is equal to $x^2 + v^2 - 2vy + y^2$, the sum of the squares of the two sides. That is to say, $x = \sqrt{s^2 - v^2 + 2vy - y^2}$ or $y = v + \sqrt{s^2 - x^2}$."[35] With this, one can eliminate x and y by means of the equation of the curve.[36] The equation, however, is yet undetermined since both v and s are still unknown; Descartes needs another condition.

Seeing that the normal through a point on a circle is the radius through the point, Descartes says that if PC is the normal through point C then the circle having PC as its radius ". . . will *touch but not cut* [emphasis added] the curve CE; but if this point P be ever so little nearer to or farther from A than it should be, this circle must cut the curve not only at C but also in another point."[37] This means that if P is not on the normal and if we have values for s and v, then eliminating, say x, from the equation $x^2 = s^2 - v^2 + 2vy - y^2$, we

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[33] *La Géometrie*, p. 342. There seems to be a mistake here. Actually $y = v - \sqrt{s^2 - x^2}$.

[34] Smith and Latham's remark here (p. 95, n. 142) that the equation to which Descartes refers is the *equation of the curve* is clearly wrong. This is evident not only from Descartes' text here, but also from the specific examples he gives afterwards.

[35] *La Géometrie*, p. 342.

[36] The mathematically minded reader will observe that this is not a trivial step. That one can isolate x or y, at least in a neighborhood of a point on the curve and under certain conditions, is given by the implicit-value theorem (see any good book on advanced calculus such as Michael Spivak's *Calculus on Manifolds*, pp. 40–42). Descartes, unjustifiably, sees no difficulty in this. What is more important to us, here and throughout, however, is the character of Descartes' approach rather than its correctness by modern standards.

[37] *La Géometrie*, pp. 345–346.

obtain an equation in y having *two roots*, y = QA and y = MA (see fig. 14), since PA still equals v and PC = PE = s. If, however, P *is* on the normal, *then these two roots coincide*, that is, the equation $x^2 = s^2 - v^2 + 2vy - y^2$, considered an equation in y, must be of the form $(y - e)^2 = y^2 - 2ey + e^2$. Herein lies the key to the method. For by comparing terms in the expressions $y^2 - 2ey + e^2$ and $y^2 - 2vy - s^2 + v^2 + x^2$ one can find equations relating e, v, and s; e can, then, be set equal to y, and since the value of y is known, C being given, the values of v and s can be found. An example is certainly in order. Descartes gives three examples. We shall look at the first.

Consider an ellipse whose equation is $x^2 = ry - \frac{r}{q}y^2$ where MA is a segment of the axis, CM an ordinate (see fig. 15), r the *latus rectum*, and q the transverse axis.[38] Descartes eliminates x from the equation of the ellipse and the equation $x^2 = s^2 - v^2 + 2vy - y^2$ to obtain $s^2 - v^2 + 2vy - y^2 = ry - \frac{r}{q}y^2$, or, $y^2 + \frac{qry - 2qvy + qv^2 - qs^2}{q - r} = 0$. Comparing the last expression to $y^2 - 2ey + e^2$, one finds $\frac{qry - 2qvy}{q - r}$ must equal $-2ey$ or $-2e = \frac{qr - 2qv}{q - r}$ while e^2 must equal $\frac{qv^2 - qs^2}{q - r}$. From the first of these equations we have $v = e - \frac{r}{q}y + \frac{1}{2}r$ so that, setting e = y, we have $v = y - \frac{r}{q}y + \frac{1}{2}r$, which is now determined since y is known. We need not go further, as Descartes points out, since v alone is enough to find the position of the normal CP.[39]

In the context of this chapter, the important observation to be made regarding Descartes's way of finding the normal to a curve is that, by reducing the problem to algebra, Descartes can speak of the circle cutting the given curve and touching the curve *in the same terms*. The whole investigation of how conic sections meet becomes, for the algebraist, an investigation of *the roots of equations*; the curves cut when the equation determining the points of intersection has two distinct roots, while they touch when the equation has two identical

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[38] For the equation, Descartes cites *Conica* I.13, consistent with our remarks regarding Descartes in the last chapter.

[39] *La Géométrie*, pp. 343–348.

roots, and one can move smoothly from cutting to touching by letting the difference between the roots vanish. A case by case investigation of the various configurations of conic sections, like that in Book IV of the *Conica*, becomes entirely superfluous, and, needless to say, there is indeed no such investigation in *La Géométrie*. Moreover, the question of the greatest number of points at which conic sections, or any other curves, can meet becomes only a question of the greatest number of roots an algebraic equation can have. This, of course, is not a trivial question, but it is one that Descartes already takes to be dependent on the degree of the equation alone, and, in Book III of *La Géométrie*, he *assumes* the answer, saying simply, "Every equation can have as many distinct roots (values of the unknown quantity) as the number of dimensions of the unknown quantity in the equation."[40] With that, Apollonius's Book IV, from the algebraic point of view, becomes a long monotonous exercise, meriting Heath's remark, quoted at the start of this chapter, that the book is ". . . on the whole dull . . ." But its results are certainly not trivial and the means of reaching them certainly not dull when one stays within the confines (which no ancient Greek could escape) of Greek geometry. Apollonius respected (he had no choice) those confines while Descartes violated them robustly and loudly.

The Possibility of Different Conic Sections in One Plane

So, the contrast between Apollonius and Descartes shows how differently they approach the problem of the meeting of curves. It also, however, brings us back to the main issue of this chapter, which is the elementary character of Book IV of the *Conica*. Up until now, we have stressed that Apollonius's goal of finding what is the greatest number of points at which conic sections and opposite sections can meet one another forces him to investigate the *mode* in which they can meet. But behind that investigation, there lies an even more basic presupposition, namely, *that conic sections can meet in one plane at all.* As in our discussion of Descartes in the last chapter, here too the question hinges on *where* things are: *where do the conic sections meet?* For the Cartesian, the answer to this is the same as the answer to

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[40] *Ibid.*, p. 372.

the question where is a point. This is so because it is the nature of analytic geometry that it *begins* with place: it has a single reference plane, the Cartesian plane, and any point in it can be given a location from the start; the truth is, giving location is *all* one can do in this plane. The Cartesian plane is an abstraction, an objectless place of places. In Apollonius's geometry, however, there is *also* a plane of reference. It is the plane that appears explicitly in the "constructions" at the close of Book I. This plane of reference, or, literally, the "plane set down" (τὸ ὑποκειμένον ἐπιπέδον) is not an abstract place of conic sections, it is the very concrete plane that produces them—it is the *cutting plane*. Conic sections are always in this plane and everything said about them is assumed to be there too. Thus, when Apollonius says in I.24, say, that "If a straight line, meeting a parabola or hyperbola at a point, when produced both ways, falls outside the section, then it will meet the diameter," he does not have to qualify his statement and say that the straight line *must be in the plane* of the parabola or hyperbola!

But once a conic section has come to be in this plane by its cutting a cone, how can *another* conic section come to be in the *same* plane, and at a given place in it? Indeed, how are we to understand the mode in which conic sections can meet, if we do not fully understand how different conic sections can come to be in a given cutting plane, in the first place? This, we believe, is the true guiding question of the propositions concluding Book I, propositions I.52–60. Recall that in I.52, for example, we are asked "to find in the plane (εὑρεῖν ἐν τῷ ἐπιπέδῳ) the section of a cone called parabola." To do this, we are given a "plane set down," a line in the plane given in position and bounded at a point, and another line given only in magnitude. The task is accomplished, as we discussed in detail in the last chapter, when we are able to conceive (νοέιν) a cone cut by the plane in such a way that a parabola is produced having the line given in position as its diameter, the bounding point as its vertex, and the line given in magnitude as its *orthia.* Proposition I.52, and the other "constructions" at the end of Book I thus complement I.11–14. For in propositions I.11–14, *first a cone is given*—and one might as well assume that it is the same cone in all four propositions—then it is shown how *different planes* can cut the cone to produce the different conic sections; the way the plane cuts the cone determines the diameter, ordinate direction, and *symptoma* of each section produced. In the sequence of propositions beginning with

I.52, on the other hand, we are first *given a plane*, a bounded line to be the diameter, and an ordinate direction—and we might as well assume that, like the cone in I.11–14, the plane is the same in all these propositions—then it is shown how *different cones* can be conceived to produce the given sections, whose positions are given by the positions of the bounded line. These propositions completing Book I, therefore, provide us with the theoretical basis for drawing *in a single plane any number of conic sections* having any diameter, vertex, and upright side. And it is by no whim, then, that the final proposition of Book I involves, *for the first time*, drawing in a plane *two* sections at once—appropriately enough, a pair of *opposite sections*, the conjugate sections.

Of course, *other* geometric entities, circles and lines, could be in the same plane as a conic section right from the start, and, of course, that assumption Apollonius had no need to elucidate, or even to state, since Euclid had done so already in the postulates to Book I of the *Elements*. This is obvious. Nevertheless, the observation helps to bring out that the question of how different conic sections can come together in a single plane is not only a concern at the end of Book I, but one, truly, throughout Book IV as well. Let us take another look at the propositions making up the first half of Book IV. In the section above, we looked at propositions IV.1–23 in some detail in order to show only how they serve to emphasize the opposite sections; we did not really discuss their purpose in Book IV taken as a whole. It is true, they are *used* in the book, but it cannot be said that their purpose is to be lemmas for other central propositions such as IV.25. This is for the simple reason that few of them are actually used after IV.25, and IV.25 itself only depends on IV.9, which depends on no other proposition in Book IV. In fact, as one goes through Book IV, one is surprised just how little they are used after IV.25: proposition IV.1 is used in IV.26 and 48, IV.9 is used in IV.25 and IV.44, and that is it. This proves, to our mind, that propositions IV.1 to 23 must have a thematic role in Book IV rather than a logical one; they must be an essential part of Apollonius's presentation of his stated theme for the book. The likely purpose of these propositions, in which only lines and single conic sections or single pairs of opposite sections appear, is to put the discussion of how conic sections and opposite sections meet one another in the context of a discussion of how *lines* meet, cut or touch, conic sections and opposite sections. There seems to be no other good

explanation as to why Apollonius should choose to take up almost half the book with these propositions, which have so little weight in the book's deductive structure.[41] The great number of these propositions and their careful development suggests also that, for Apollonius, the main part of the book needed more preparation than what could be provided for by the last propositions in Book I, in which only the possibility of conic sections meeting is established; Book IV, the final book among the elements of conic sections, shows that that possibility requires serious and extended reflection to be properly understood.

Conclusion

The two reasons given in this chapter for why Apollonius includes Book IV in the elements of conic sections both refer to the geometric origins of the conic sections. Book IV's emphasis on the opposite sections, an emphasis that to a greater or lesser degree is to be found in all the elementary books, turns our attention once again to their peculiar nature and, especially, to the double conical surface, which alone can produce them. But the theme of the book itself, the greatest number of points at which conic sections and opposite sections can meet one another, forces us to reflect on how conic sections can meet, which, in turn, forces us to reflect on how different conic sections can arise in a given plane. This latter point, brings us back to the propositions in which the conic sections are first produced, I.11–14, and to their complementary propositions at the end of Book I.

Clearly, mathematical elements are not necessarily featureless atoms with which things are built, but, as we discussed in our considerations of Books I to III, they are such things that contain necessary ideas for understanding the whole subject. They can do this by being

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[41] That these propositions "prove the converse of the propositions in Book III [III.37–40] about the harmonic properties of the pole and polar for a large number of particular cases" (Heath, *History*, II.157) gives us no light on the question. For not one of the propositions IV.1–23 depends on any proposition in Book III after III.39, so that if proving those converses were the *purpose* of IV.1–23, one could expect to find them in the middle of Book III; one would not expect to find them at the beginning of Book IV, whose stated goal has little to do with "the harmonic properties of the pole and polar," notions, needless to say, entirely absent from Hellenistic mathematics.

themselves starting points, propositions or objects on which everything else logically depends, or by forcing the reader to dwell on the foundations of the subject at hand. Book IV of the Conica, it seems, is a book of elements in this latter sense. Together, Books I to IV, while providing us with tools for further study, engage us in an inquiry as to what conic sections are and how they come to be.

We must now turn to the other books of the *Conica*, the non-elementary books. In the next three chapters, we shall consider each of the Books V to VII in depth. However, what has been said so far about the elementary books must be kept in mind throughout, and, later, we shall have an opportunity to reconsider the elementary books in the context of the whole of the *Conica.*

Figures

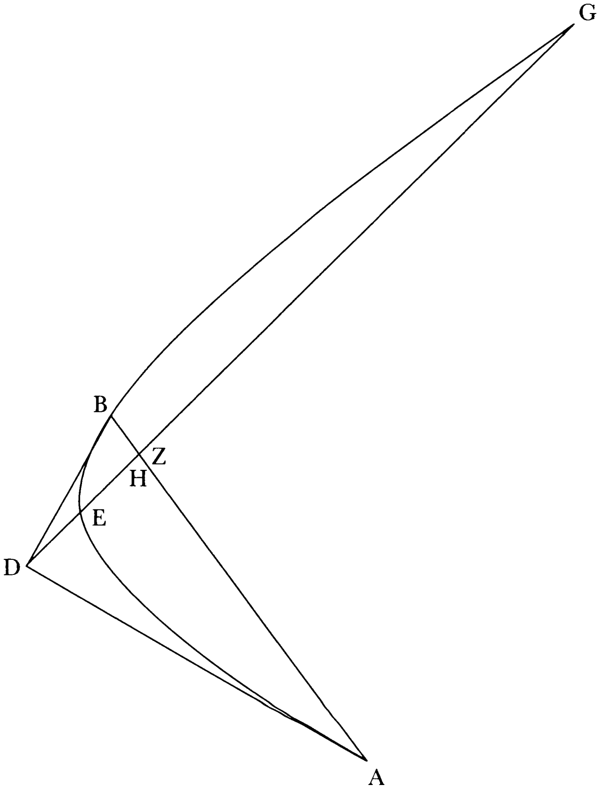

Fig. 1

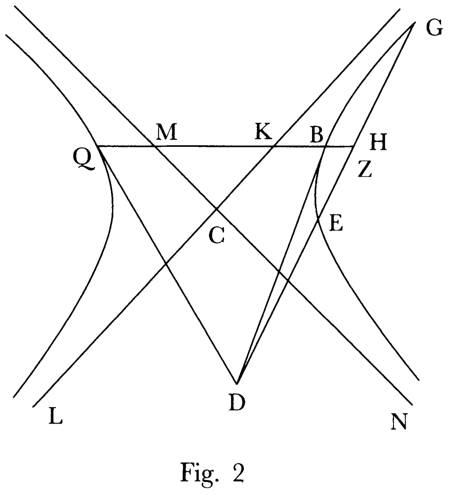

Fig. 2

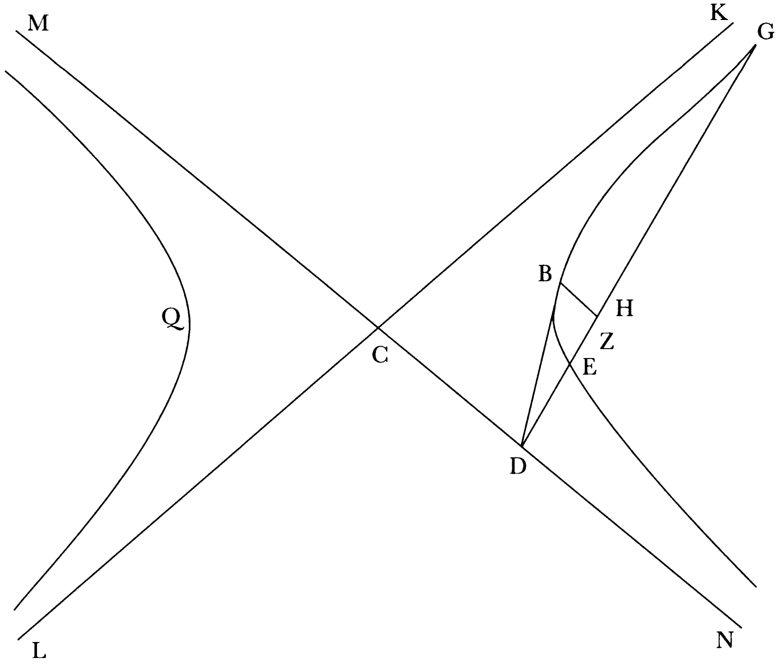

Fig. 3

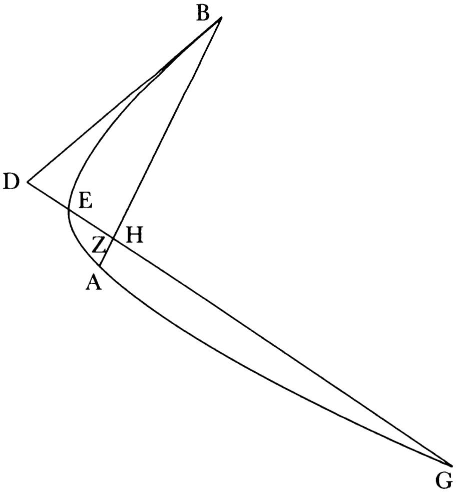

Fig. 4

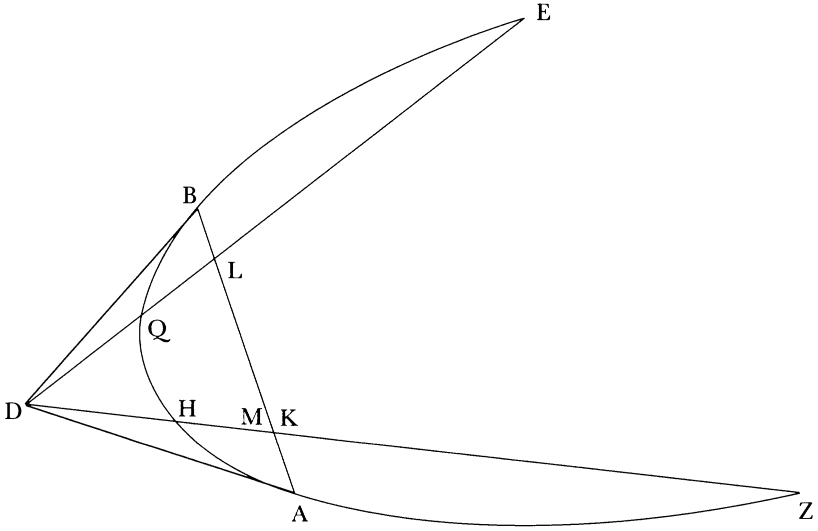

Fig. 5

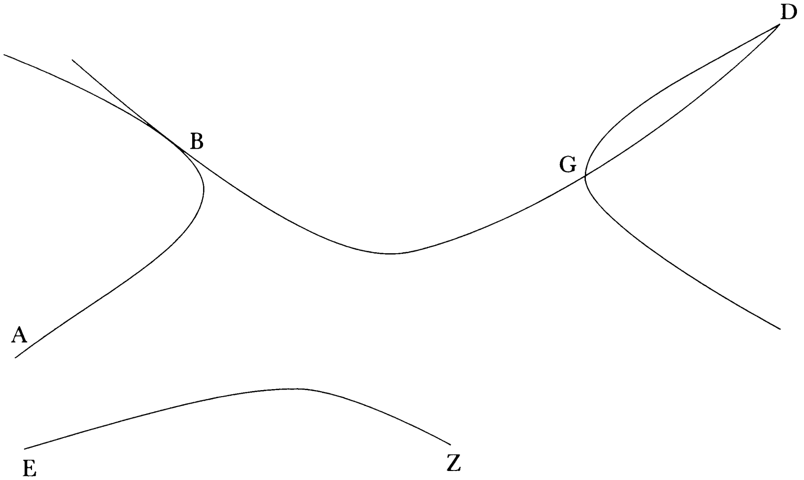

Fig. 6

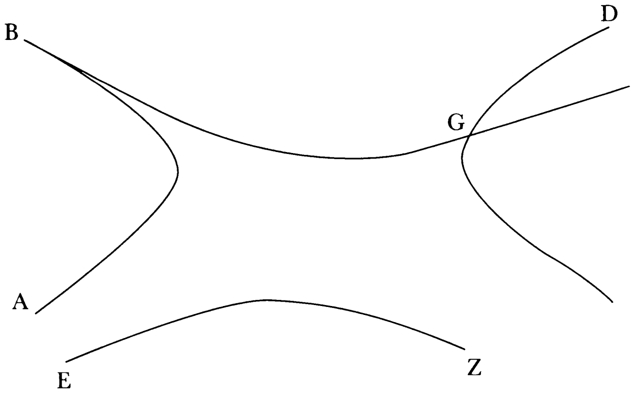

Fig. 7

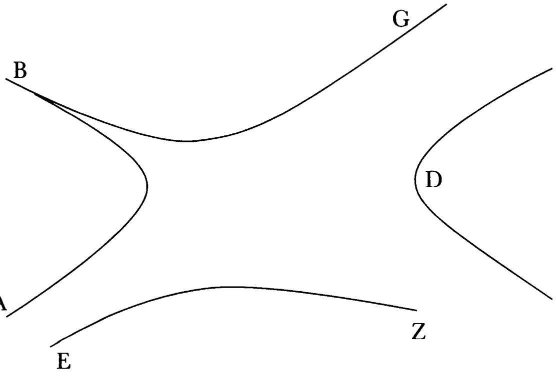

Fig. 8

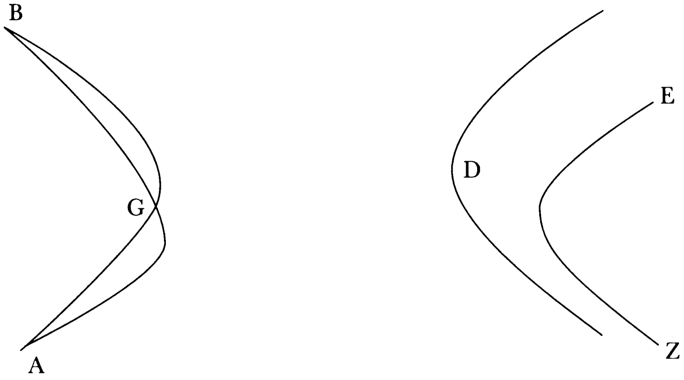

Fig. 9

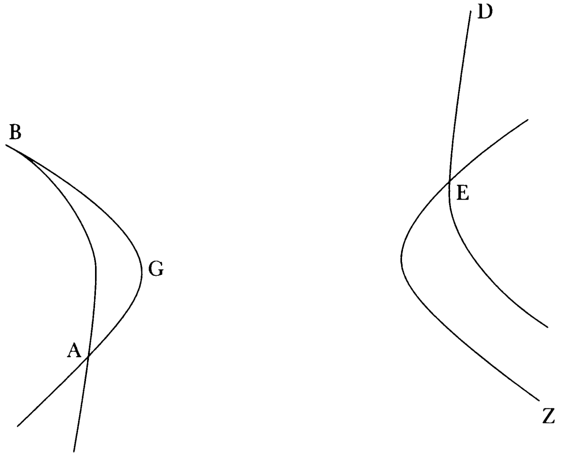

Fig. 10

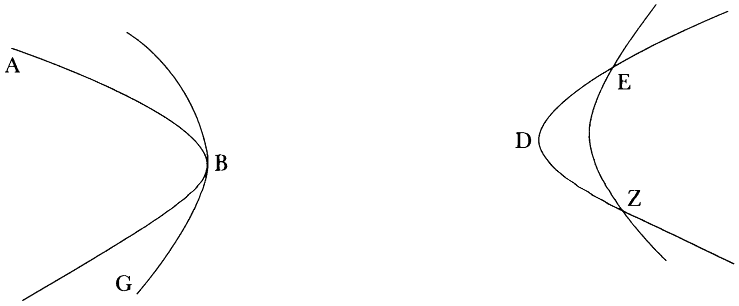

Fig. 11

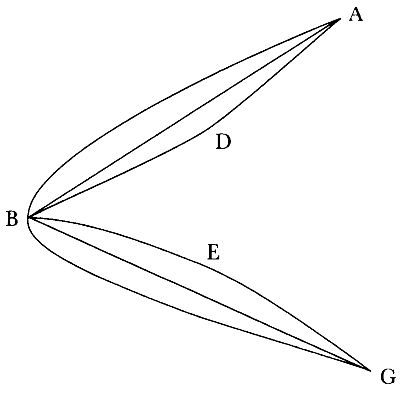

Fig. 12

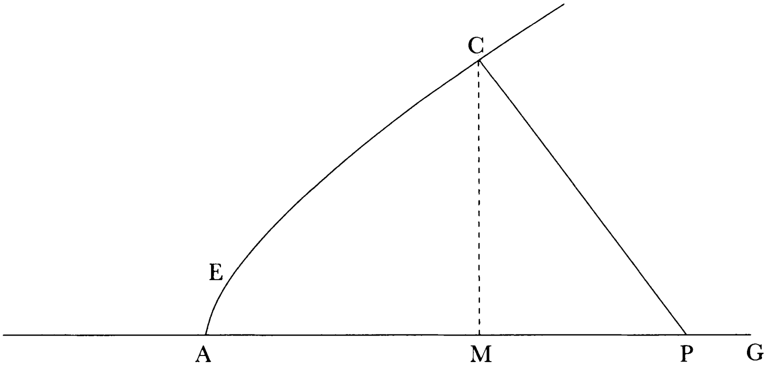

Fig. 13

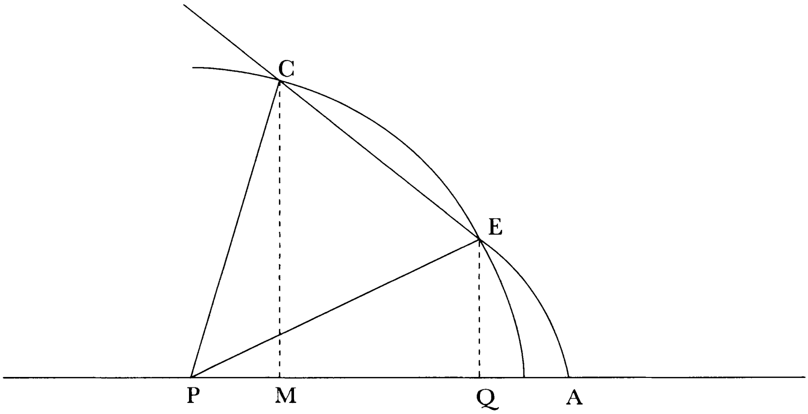

Fig. 14

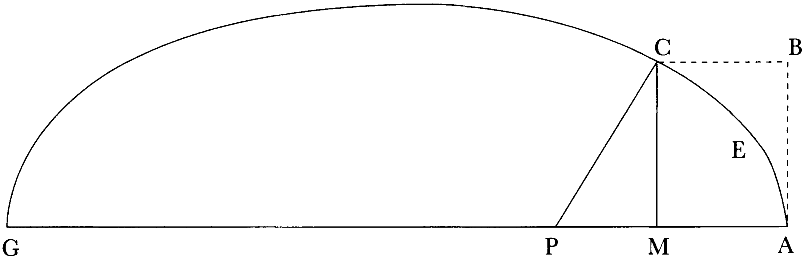

Fig. 15

CHAPTER FOUR

MAXIMUM AND MINIMUM LINES: BOOK V OF THE *CONICA*

Introduction

With its seventy-seven propositions, Book V is the longest book in the *Conica*. It is also, probably, its most celebrated book.[1] Thus, Toomer notes, "Of Books V–VII, which are, to judge from Apollonius' own account, largely original, Book V is that which has particularly evoked the admiration of modern mathematicians . . .," and Heath calls the book ". . . a veritable geometrical *tour de force*."[2] Like the other books of the *Conica*, Book V is tightly organized, particularly thorough,[3] and deep. Our intention is not to deny the praise the book has so often received, rather, to see that such praise is well placed. The problem is that the admiration for Book V is always based, one way or another, on the understanding that the book concerns *normals* to conic sections and that its great achievment is the determination of the envelope of these normals, the *evolute* of the curves. This understanding is hardly ever questioned and is usually stated as if it were plain fact. Boyer writes, for example, "the books of the second half of the *Conics* include a study of normals approaching the modern theory of evolutes . . .," or Toomer, ". . . [Book V] deals with normals to conics, when drawn as maximum and minimum straight lines from particular points or sets of points to the curve. Apollonius finally proves, in effect, that there exists on either side of the axis of a conic a series of points from which one can draw only one normal to the opposite side of the curve, and shows how to construct such points: these points form the curve known, in modern terms, as the *evolute* of the conic in question," or Struik,

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[1] We shall return to this theme in the next chapter, for there the question will arise, why, by contrast, Book VI is the *least* celebrated of the books in the *Conics*.

[2] G. J. Toomer, "Apollonius of Perga" in *DSB*, I, 187 and T. Heath, *A History of Greek Mathematics*, II, 159.

[3] As Toomer remarks, it is ". . . the most exhaustive treatment of a single topic in conics from antiquity" (*Apollonius* Conics *Books V to VII*, p. xxxix).

"[The *Conics*] is a treatise on the ellipse, parabola, and hyperbola, introduced as sections of a circular cone, and penetrates as far as the discussion of the evolutes of a conic."[4] Yet, despite the confidence expressed in these statements, it is hard to find unambiguous support in the text for normals being Apollonius's main concern, let alone the concomitant subject of evolutes. For one, Apollonius says nothing about normals in the general preface at the start of Book I nor in the prefatory letter to Book V itself. Instead, Apollonius speaks about *minimum and maximum lines.* It is true, one must grant, that in the letter opening Book V Apollonius does speak about *tangents*, and this might be construed as reference to normals since normals are, by definition,[5] lines perpendicular to the tangents, but what he says about tangents is, essentially, that they are not to be the main subject of the book. Moreover, the general flow and arrangement of propositions in Book V, as we shall see, do not point to normals as a central theme. Indeed, the propositions explicitly about minimum and maximum lines being perpendicular to tangents are relatively few and seem to have the character more of lemmas than of main theorems. For these reasons, the view that "Apollonius' theorems on maxima and minima are in reality theorems on tangents and normals to conic sections,"[6] needs reexamination. Such reexamination, then, is the work of the first half of this chapter. We shall consider critically the evidence that might be put forth to support the conventional view that the subject of Book V is the properties, determination, and construction of normals to conic sections. This "evidence" includes the presence of the "normal propositions" V.27–33, the

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[4] C. B. Boyer, "Analytic Geometry in the Alexandrian Age," p. 146; Toomer, "Apollonius of Perga," p. 187; D. J. Struik, *A Concise History of Mathematics*, p. 66. For other examples, see H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, pp. 284–309 and Heath, *A History of Greek Mathematics*, II, 159. Even Knorr, while he has serious reservations about Apollonius's determining the evolute, assumes that the focus of Book V is the study and construction of normals (see, *The Ancient Tradition of Geometric Problems*, pp. 313–321). In this connection, we should say that in his *Apollonius* Conics *Books V to VII*, written somewhat later than the *DSB* article, Toomer rejects this view, saying, "Apollonius, as his preface . . . implies, considers the topic of the book to be *minima* and *maxima* to conics, and not as *normals* to conics (the term by which Heath's version usually refers to them, misleadingly)" (p. xxxix).

[5] We should point out that there is, to our knowledge, no general definition of a *normal to a curve*, as there is none for a *tangent* to a curve, in the Hellenistic texts that have come down to us. So, when we say, "by definition," this is really "by *our* definition."

[6] Boyer, *A History of Mathematics*, p. 169.

application of "normals" in optical studies such as Diocles' *On Burning Mirrors*, the ease with which one can (with modern means, of course) use V.51–54 to derive the equations of the evolutes, and the apparent obviation of difficulties concerning the very notion of minimum and maximum lines, in particular, that minimum and maximum lines are not always in every respect shortest and longest lines.

It may appear strange that in this chapter we should argue against an interpretation of Book V involving normals to conic sections having argued so carefully and emphatically in the last chapter for the geometrical basis and outlook in the "elements" of conics, Books I–IV. For is not the normal to a curve a geometrical entity? Yes, it is, but as we have said before, this work will have to show not only *that* the *Conica* is built on a geometrical foundation and developed in that spirit but also *in what way* it is geometrical. Indeed, all the chapters of this work treat both these aspects with greater or lesser emphasis. Here, however, the question of *what* is the geometry of Apollonius will be somewhat more urgent than whether it *is* geometry. Accordingly, in the second half of this chapter we shall try to understand what are exactly minimum and maximum lines and what it means to find them. The first step here will be to consider propositions in Book III of Euclid's *Elements* analogous to Apollonius's propositions in Book V. This is not to make the claim that Apollonius is consciously modeling Book V on Euclid's work (though we do not exclude the possibility), but to widen the setting within which we can think about Book V. Following this, we shall suggest that if one were to categorize *the kind* of investigation seen in this book of the *Conica*, one would have to put it near investigations such as that carried out by Apollonius and others into *neusis*. Of course, what we see in Book V is *not neusis*, so we shall call it only *neusis-like*. This suggestion, if it proves to be more than mere speculation, will place the work of Book V in line with a *specifically* Greek inquiry.

The Conventional View—Normals as Maxima and Minima

Let us begin, then, by restating the conventional view, which we shall do with the help of Heath's version of the *Conica*. This version, as mentioned in the prologue, reworks the original to make it more congenial to a modern mathematician. In general, this is the kind of thing that makes Heath's work of dubious value for the historian,

revealing more of Heath's mind than that of Apollonius. In this section, however, since we *are* interested in Heath's view, let us take advantage of Heath's desire that his version ". . . should exhibit under different headings the successive divisions of the subject, so that the definite scheme followed by the author may be seen as a whole."[7] So, what, according to Heath, is the "scheme" for Book V of the *Conica*? Here are the "headings": "Normals as Maxima and Minima," "Propositions Leading Immediately to the Determination of the *Evolute*," "Construction of Normals," and "Other Propositions Respecting Maxima and Minima." Among different writers on the *Conica* there are, of course, slight differences in emphasis concerning this "scheme." For example, judging by remarks made elsewhere,[8] Heath thinks that the second section is the high point, while Zeuthen emphasizes the "Construction of Normals," seeing this as an example of a "solid" construction.[9] But such variations aside, we think it is fair to say that Heath's "scheme" makes the nexus of ideas particularly clear, and is representative enough of what we have called the conventional view, to let it stand as our model of this view.

The division entitled "Normals as Maxima and Minima" in Heath's "scheme" covers propositions V.1–50. Among these, only *seven* propositions, V.27–33, are in any way explicitly about minimum and maximum lines being perpendicular to the tangent. Yet, Heath still sees *all* of these propositions relating to normals as maxima and minima. This is hard to swallow. Consider, for example, V.8. Apollonius writes,

> If a point is marked on the axis of a parabola, the distance of which from the vertex of the section is greater than half the *latus rectum*, and there is cut off on the axis, from that point which was marked on it, towards the vertex of the section, a line equal to half the *latus rectum*, and from the [other] end of that line which was cut off there is drawn a perpendicular to the axis, and that perpendicular is extended to meet the section, and there is drawn from the place where it meets the

[7] Heath, *Apollonius of Perga: Treatise on Conic Sections*, p. ix.

[8] *A History of Greek Mathematics*, II, 159.

[9] Zeuthen's chapter on Book V in *Die Lehre von den Kegelschnitten im Altertum* comes in the course of a discussion of solid problems and begins "Außer den bereits angeführten [the allusion is to solid problems involving *neusis* solved by means of conics] ist nur noch ein Beispiel dafür aufbewahrt, wie die Alten eine körperliche Aufgabe behandelten, nämlich Apollonius' Konstruktion einer Normale von einem gegebenen Punkt an einen gegebenen Kegelschnitt" (p. 284).

> section a line to the point which had been marked, then that line is the shortest of the lines drawn from that point which was marked on the axis to the section, and of all the other lines on both sides [of it], those closer to it are shorter than those farther [from it]; and the square on each of them exceeds the square on the shortest line by an amount equal to the square on the line between the foot of the perpendiculars to the axis from the two of them.[10]

In Heath's version, however, V.8 begins as follows:

> In a parabola, if G be a point on the axis [see fig. 1] such that AG > $\frac{1}{2} p_a$, and if N be taken between A and G such that
>
> $$NG = \frac{p_a}{2},$$

then, if NP is drawn perpendicular to the axis meeting the curve in P, PG is the *minimum* straight line from G to the curve [*or the normal at P*] [emphasis added].

> If *P′* be any other point on the curve, *P′G* increases as *P′* moves further from *P* in either direction.
>
> Also
>
> $$P'G^2 = PG^2 + NN'^2.$$[11]

So, from the first part of Heath's statement,[12] taken literally, we see that Heath genuinely understands V.8 to be saying that given a point on the axis of a parabola further from the vertex than half the *latus rectum*, one can determine a *normal* from the given point to the parabola in the way described (taking NG equal to half the *latus rectum, etc.*). Of course, he *must* set in brackets the clarification that PG is the normal, *because this appears nowhere in Apollonius's enunciation*; in *Apollonius's* enunciation there is no mention of any line perpendicular to a tangent nor, for that matter, any mention of a tangent to a parabola, only of lines from a point on the axis to the parabola being longer or shorter than other such lines. That this latter theme,

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[10] As always, unless noted otherwise, all quotations from Book V of the *Conica* are from *Apollonius* Conics *Books V to VII* by G. J. Toomer. Additions in square brackets, with the exception of those indicating figures, are Toomer's. Where the additions are our own, we shall follow the brackets with an asterisk.

[11] Heath, *Apollonius*, pp. 143–4.

[12] We shall return to the form of the second part of the statement, that "If *P′* be any other point on the curve, *P′G* increases as *P′* moves further from *P* in either direction," later.

and only this, is the theme of V.8 can be seen right at the start of the proof:

> Let there be a parabola ΑΒΓ [see fig. 2], with axis ΓΔ, and let line ΓΕ be longer than half the *latus rectum*, and let half the *latus rectum* be ΖΕ. We draw perpendicular ΖΗ to ΓΕ, and join line ΕΗ.
>
> Then I say that line ΕΗ is the shortest of the lines drawn from point Ε to the section.
>
> And as for the other lines drawn from ΑΒΓ [to point Ε], those of them closer to line ΕΗ are shorter than those farther [from it], on both sides [of ΕΗ].
>
> So we draw from point Ε to the section lines ΕΚ, ΕΛ, ΕΘ, ΕΑ. Then I say also that the square on each of these exceeds the square on ΕΗ by an amount equal to the square on the line between the foot of the perpendicular from it [Κ, Λ, Θ, Α] and point Ζ.

Thus, neither in the enunciation, nor in the setting out, nor in the construction up to this point do we see anything other than lines from point Ε to the parabola ΑΒΓ. Moreover, these lines are to be compared only among themselves, and that, only with respect to magnitude. The rest of the proof has no surprising turns and at the end of it what we have learned is that ΕΗ is the shortest line connecting Ε and a point on the section; we have learned exactly how it differs from other lines drawn from Ε, and, from that, that the closer a line from Ε is to ΕΗ the smaller it is. In fact, the proof only *proves* how ΕΗ differs from other lines drawn from Ε, which in Heath's enunciation seems only an afterthought, "*Also* [emphasis added] $P'G^2 = PG^2 + NN'^2$" (although here we must say that, in his own enunciation, Apollonius is not much more emphatic than Heath is, as far as we can tell). Yet, this property itself has nothing to do with any property of the tangent that Apollonius has expounded elsewhere in the *Conica*. The proof, therefore, aims only at the relative size of the segments between Ε and the parabola and on the position of the segments to the extent that they have Ε as an endpoint and that they are closer or farther from ΕΗ.

Similar remarks could have been made about any one of the first twenty or so propositions in Book V. Apollonius's text, therefore, has given its readers, reading up to, say V.8, no reason to think that the minimum lines have, in general, some connection with normals. Perhaps, however, the deductive structure of the book *demands* Apollonius's developing a theme of normals in this way, that is, that V.8, among others, only set the stage for the "normal propositions"

V.27–33. There is something to this claim. For V.27, proving that "The line drawn from the end of one of the minimum lines we mentioned tangent to the section is perpendicular to the miminum line" in the case of the parabola, *relies* on the partial converse to V.8, proposition V.13.[13] But this does not mean that Apollonius could not prove V.27 in any other way, nor does it mean necessarily that V.27 is the goal of V.13, rather than an application, a mere corollary to V.13. Indeed, consider V.29. Following V.28, which is the proposition that continues V.27 for the cases of the hyperbola and ellipse (and which, we should add, relies on V.14–15, the propositions analogous to V.13), we have "That [the line drawn from the end of one of the minimum lines tangent to the section is perpendicular to the minimum line]* may be proven in another way: that is as follows: I make the conic section ΑΓ and its axis ΒΔ [see fig. 3].

> Then I say that the line drawn from the end of the minimum line tangent to the section is perpendicular to the minimum line.
> Let the minimum line be ΑΒ and the tangent ΑΔ.
> Then I say that <ΔΑΒ is right.
>
> [*Proof*:]
>
> For if that is not so, we draw perpendicular ΒΕ to line ΑΔ.
> Then ΑΒ > ΒΕ.
> So how much the greater is it [ΑΒ] than ΒΖ.
> [But] that is absurd, for ΑΒ is the minimum line.
> Therefore, <ΔΑΒ is right.

Regarding this simple, almost trivial, proposition, one notes immediately that although ΑΓ is said to be a conic section and ΒΔ its axis, these assumptions are unnecessary for the proof, except, perhaps, to know that there is, in fact, a tangent and a minimum line at point A. As for AB, all one needs to know is *that* it is a minimum and nothing else.[14] So *if* Apollonius wanted us to associate

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[13] Specifically, V.13 states: "If there is drawn from a point on the axis of a parabola the minimum of the lines drawn from that point to the section so as to form an angle with the axis, then that angle which it forms with the axis will be acute; and if a perpendicular is dropped from its [other] end to the axis, then [that perpendicular] cuts off from it [the axis] a portion equal to half the *latus rectum*."

[14] One can prove the case in which AB is a *maximum* line similarly: if the tangent ΑΔ is *not* perpendicular to the maximum line AB, let AX be perpendicular to AB (see fig. 4). Then AX falls within the section. Let X be on the part of AX that is in the section. Join BX and extend it to point P, where BX cuts the section (note that it *does* cut the section since ΑΔ falls outside the section). Then BX is longer

minimum lines always with normals, he *could have begun* the book with V.29, or, at least, have placed it towards the beginning of the book.[15] For example, he might have inserted it just before V.8, for in each of the previous propositions the minimum line is a segment of the axis ending at the vertex, and is, therefore, a normal since the tangent at the vertex (with respect to an axis) is perpendicular to the axis. V.8 introduces the first minimum line that is not a segment of the axis, so V.29 preceding V.8 would have shown that such minima are, as in the previous cases, also normals. But, of course, Apollonius does not proceed in this way. One possible explanation is that V.29 is the work of an interpolator. This is, for example, Toomer's opinion: "I doubt very much whether this curious proof (which appears to be inspired by V.31) is by Apollonius, and suspect that it is an interpolation in the Arabic tradition."[16] Indeed, the claim is strengthened by the rarity in Greek mathematical texts

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than BA, and BP is longer than BX. So BP is longer than BA, contradicting the assumption that AB is a maximum. Apollonius, however, does not argue this way. Apollonius proves that the tangent drawn from the end of a maximum line is perpendicular to the maximum line in V.30, where he proves it by means of V.23. V.23 allows him to reduce the case of a maximum line to one of a minimum line, and it is very likely he wanted to show the strength of V.23 since, according to Apollonius's prefatory letter, maximum lines were not treated by his predecessors.

[15] Thus, as a review of the main results in Book V, J. Itard (who fully accepts the book's being about "normals") begins the "Introduction mathématique" in his "L'angle de contingence chez Borelli: Commentaire du livre V des Coniques d'Apollonius" (in *Essais D'Histoire des Mathématiques*, p. 118) with what is, essentially, a restatement of V.29. Establishing *from the start* that the tangent at the end of a minimum (or maximum) line must be perpendicular to the minimum (or maximum) line gives these results a direction that is attenuated, if not lost, in Apollonius's version where the proposition appears well into the book. We should stress here, that the *mere fact* that proposition V.29 is stated in terms of specific objects, conic sections, while its proof could be applied generally, is not the *main* point; indeed, there are other instances in Greek mathematics of such propositions. For example, *Elem.*,IV.13 states: "*In a given pentagon* [emphasis added], which is equilateral and equiangular, to inscribe a circle," while the proof, in fact, is applicable to *any* polygon that is equilateral and equiangular and depends only on propositions from Books I and III. Like V.29 in Book V of the *Conica*, therefore, *Elem.*,IV.13 could be placed at the beginning of *Elem.*,IV. But what effect would this have? It would not help to emphasize the inscribing and circumscribing of figures as a focus of the book, for that is done well enough by the initial definitions of Book IV, and it would, perhaps, weaken the sequence of propositions about the pentagon, whose construction in *Elem.*,IV.11 is certainly an impressive highlight of the book. It is in the effect of its repositioning that the case of *Conica* V.29 stands out, for, unlike *Elem.*,IV.13, the repositioning of V.29 to the beginning of Book V would have a *crucial* influence on the thematic structure of the book.

[16] Toomer, *Apollonius* Conics *Books V to VII*, p. 559. See also p. xlvi in the same work.

of an *alternate* proof for a proposition by the *same* mathematician in the *same* work.[17] However, even if it is true that V.29 is an interpolation, and it may well *be* true, this alone gives only a partial and not very satisfying answer to the question at hand, namely, why Apollonius did not proceed in a manner like the one outlined above if his goal in V.1–50 was, in fact, to consider normals as minimum and maximum lines. The conclusion that Apollonius constructed the book as he did because he did not know or could not prove a result such as V.29 is not convincing. Clearly, it was well within Apollonius's powers to find and prove V.29. Indeed, as Toomer remarks parenthetically in the passage quoted above, the method of proof in V.29 is very similar to that in V.31, the converse to V.27–29. Furthermore, outside the *Conica*, one also finds a precursor for V.29 with which, it is reasonable to assume, Apollonius was familiar. *Elem.*,III.18 states, "If a straight line touch a circle, and a straight line be joined from the centre to the point of contact the straight line so joined will be perpendicular to the tangent." Paraphrased, Euclid's proof runs as follows: let DE be tangent at the point C to circle ABC whose center is F (see fig. 5). Join FC. Then FC must be perpendicular to DE. For if not, draw FG perpendicular to DE. Then, since <FGC is right and, hence, <FCG is acute, FC must be greater than FG. But FC is equal to FB, so FB is greater than FG, which is impossible. Therefore, FG is not perpendicular to DE, as can be proved for any other straight line except FC. Therefore, FC is perpendicular to DE. From all this, one can conclude more safely, we think, that if V.29 did not appear in the original version of Book V, it was because Apollonius wanted to prove V.27–28 *specifically* in terms of minimum lines, that is, V.27–28 were to be corollaries of V.11–13 rather than the latter being lemmas for V.27–28.

To summarize, the propositions in Book V up until the sequence V.27–33 speak of lines perpendicular to the tangent, that is, normals, neither explicitly nor implicitly. Moreover, V.29, which does not depend on any of the previous propositions and could have been stated at the outset, contradicts any claim that logical considerations require that propositions such as V.11–13 *precede* V.27–33. This suggests that up to V.33, at least, the dominating theme is minima and

[17] We have not done a thorough search on this point, but it is surely true of Apollonius and Euclid at least.

maxima and not normals, consistent with Apollonius's own statement in the opening letter. Hence, it seems thus far that Heath would have done better calling V.27–33 "Maxima and Minima as Normals," rather than V.1–50, "Normals as Maxima and Minima." But more needs to be said. For, perhaps, while not explicitly constructing the book around the theme of normals, Apollonius has normals in the back of his mind, so to speak. So let us consider the possible importance of normals for Apollonius.

The Importance of Normals

The importance of the normal to a curve (and certainly to a surface) for Heath, or for any modern mathematician, can hardly be exaggerated. Its connection with the *curvature* of a curve, and, in general, its mediating place in the *tangent-normal-binormal* triad, makes normals basic to any modern understanding of the differential geometry of curves. In this, one sees, also, the importance of the *evolute*, which can be viewed alternatively as the locus of the centers of curvature or as the envelope of the normals. To these one can add the many physical interpretations and applications of the normal to a curve and surface, in optics, for example. Indeed, we tend to believe that it is for *these* reasons that writers on Greek mathematics, such as Heath, have been all too willing to see Book V as a book on normals. However, it is not this that we wish to explore here, but only whether, within the context of the *Conica* at least, Apollonius *also* sees this subject of normals, in any sense, as a central issue.

Of course, we have already given a partial answer to this question in the last section, having shown there that it is far from clear that the propositions preceding V.27–33 have maximum and minimum lines *as normals* as their goal. The truth is that the seven "normal propositions" seem *isolated* among the other propositions of the book, a kind of detour; they do not appear to *modify* the propositions that follow or to be used in any *pivotal* way,[18] except, perhaps, in V.34, which might, in any case, be seen as the *eighth* proposition in the group. Propositions V.44–52, which, as Toomer rightly says,

[18] This is not to say that they are not used at all. Besides V.34, they are used and, as far as we can see, needed in V.64,72,73. We shall consider V.64 below.

"constitute the core of the book,"[19] appear, accordingly, to continue the line of thought in V.4–26 rather than that in V.27–33. V.44, for example, states: "If two of the minimum lines are drawn from the axis of one of the conic sections,[20] and extended until they meet, and another line is drawn from their intersection cutting the axis and ending at the section, then the part of it [the line] falling between the section and the axis is not one of the *minima*, and if the line which was drawn is not between the two *minima*, and a minimum line is drawn from the point at which it reaches the section, then [that minimum line] cuts off from the axis adjacent to the vertex of the section a line greater than that cut off by the line which was drawn; but if the line which was drawn is between the two *minima*, then the minimum line drawn from the point at which it reaches [the section] cuts off from the axis adjacent to the vertex of the section a portion smaller than the portion cut off [by the line which was drawn]; and, in the case of the ellipse, the above holds when the two *minima* and the line which is drawn all cut one and the same half of the two halves of the major axis." Here, as in V.8 discussed above, there is no mention at all of the tangent and, therefore, of the minimum line being perpendicular to the tangent. The conic section in V.44 is, again, not that to which the minimum is normal, but only that that cuts off segments from lines drawn from a point and intersecting the axis. And, as in V.8, what is spoken of in V.44 is the relationship of segments thus cut off and other segments that are *minima in the sense of* V.8 or, equivalently, V.13. Consider the first case in V.44.

With parabola ΑΒΓ (see fig. 6), axis ΔΗ, and minima ΒΖ and ΓΕ meeting at point Ο, a line ΟΛΚ is drawn from point Ο falling outside ΟΓ and ΟΒ and cutting the axis at point Λ and the section at point Κ, so that ΛΚ is the segment of ΟΛΚ falling between the section and the axis. Apollonius must prove "that ΛΚ is not one of the minimum lines, and that the minimum line which is drawn from point Κ cuts off from the axis next to the vertex of the section, which is point Δ, a line longer than line ΔΛ." Line ΚΜ is drawn from Κ perpendicularly to the axis. The whole proof of this case is then built to show that ΜΛ is less than half the *latus rectum*, so that,

[19] Toomer, *Apollonius* Conica *Books V to VII*, p. xlix.

[20] The proof in the case of the parabola is given in V.44 itself, while the proof in the cases of the hyperbola and the ellipse is given in V.45.

by V.13, KΛ is not a minimum, and that if a minimum were drawn from K it would fall to the right of Λ. Hence, the *test* for whether or not a line from a point on the axis of a parabola to a point on a parabola is a minimum line has to do *only* with whether the segment from the point on the axis to the foot of the perpendicular to the axis from the point on the parabola is, or is not, equal to half the *latus rectum*. Apollonius does not ask whether the line KΛ is or is not perpendicular to the tangent drawn at K and, having shown that KΛ is not a minimum, he does not ask whether the minimum from K makes a lesser or greater angle with the tangent than does KΛ. In other words, the test for minimality is drawn from the series of propositions V.8–15, which have nothing to do with "normalcy," rather than from V.27–33, which do.

What we have said about V.44 can be seen even more trenchantly in V.51, for here there *is* a tangent drawn at the end of a minimum line. Following a line of thought begun in V.49, Apollonius says in V.51, "But if the perpendicular [drawn from the axis]* we mentioned [in V.49]* cuts off from the axis a segment greater than half the *latus rectum*, then I say that it is possible to generate a line such that when the perpendicular drawn is measured against it, . . .[2], if the perpendicular is equal to the generated line, then it is possible to draw from its end only one line such that the part of it cut off [by the axis] is one of the *minima*; and [in that case] the *minima* drawn from the ends of the other lines drawn from that point cut off from the axis adjacent to the vertex of the section lines greater than those cut off by the lines themselves . . ."[21] The gist of the proof is as follows. Let ABΓ be a parabola and ΓZ its axis [see fig. 7]. Draw EZ perpendicular to the axis and such that the part cut off from the axis, ΓZ, is greater than half the *latus rectum*. Since ΓZ, is greater than half the *latus rectum* let ZH be *equal* to half the *latus rectum*. Cut ΓH in Θ such that ΘH is twice ΘΓ, draw ΘB perpendicular to the axis, and find some line K such that K:BΘ::ΘH:HZ. With K as the "generated line," we let EZ equal K and we need to prove that there is only one line from point E such that the axis cuts off from it a minimum. Join EB and let EB cut the axis at Λ. Then ΘH:HZ::EZ:BΘ. But EZ:BΘ::ZΛ:ΛΘ, so that ΘH:HZ::ZΛ:ΛΘ, or,

[21] In case [1] the generated line is less than the perpendicular and in case [3] the perpendicular is less than the generated line. We shall return to this proposition later, when we discuss Apollonius and the *evolute*.

componendo, ΘZ:HZ::ZΘ:ΛΘ. Therefore, ZH equals ΛΘ. Hence, ΛΘ equals half the *latus rectum*, and, therefore, ΛB is one of the *minima* by V.8. To show that besides ΛB no other line from E will have a minimum line cut off by the axis, draw MIE, draw NMΞ through M and perpendicular to the axis, and draw BO tangent to the section at B. Relying on the crucial property of the tangent to a parabola proven in I.35, namely, that the segment of the axis between the tangent and the vertex, ΓO, equals the segment of the axis between the vertex and the ordinate drawn from the point of tangency, ΓΘ, Apollonius shows that NI must be less than the *latus rectum.* But by V.13, this means that MI cannot be a minimum, and the minimum to M must fall to the right of I. Similarly, one can show also that a line such as EPA is not a minimum line. So, in this proof, unlike in V.44, we *do* have a minimum line, ΛB, *and* the tangent, BO, drawn at its end. Yet, with that, Apollonius never reminds us of ΛB being perpendicular to BO, nor does he need to, since the step where he uses the tangent does not rely on its being perpendicular to the minimum line. What we *do* need are, as in V.44, propositions V.8 and V.13, so again it is *these* propositions that we are to call up when we think of a minimum line, and *not* V.27–33, even in such an instance when the minimum line and tangent are both set before us.

Besides V.34, mentioned above, there are, however, three places in Book V where V.27–33 are cited explicitly. These are V.64,72,73.[22] Of these, we shall consider only V.64, since the latter two propositions rely on V.64 and do not use V.27–33 in a significantly different way than V.64. The proposition states, "If a point is marked below the axis of a parabola or hyperbola, such that the line drawn from it to the vertex of the section forms with the axis an acute angle, and [such that] it is not possible to draw from that point to the section a line such that the part of it falling between the section and the axis is one of the *minima*; or if only one of the lines drawn from that point to one side [of the axis], which is different from the side on which the point is, can have cut off from it [between axis and section] a minimum line: then the line drawn from that point to the vertex of the section is the shortest of the lines drawn from that point to that side of the section, and, of the remaining lines, those closer to it [the first line] are shorter than those farther [from it]."

[22] One might say they are also used in V.65–67 indirectly as these rely on the method of V.64, and it is in this method that we find the application of V.27–33.

The proof in V.64 covers only the case of the parabola and only when *no* line from the point has a minimum line cut off between the axis and the section; the remaining cases are covered in V.65–67. Thus, Apollonius writes "Let the section be, first, a parabola, ABΓ [see fig. 8], with axis AE, and let there be a point below the axis, namely Z, and let <ZAE, which is formed by the line drawn from point Z to the vertex of the section, ZA, and the axis, AE, be an acute angle, and, first, let it not be possible to draw from point Z to the section any line such that the part of it cut off between the section and the axis is one of the *minima*.

> Then I say that the shortest of the lines drawn from point Z to section AΓ is line AZ, and that, of the remaining lines [drawn from Z to the section], those closer to it [AZ] are shorter than those farther [from it].

From here, Apollonius first determines, by means of V.51, the position of Z so that, indeed, no line can be drawn from Z having a minimum line cut off between the axis and the section. By the length of this part of the proof, almost two pages in Toomer's edition and roughly half the total length of the proof, one wonders whether the application of V.51 (and of V.49 to a lesser extent) is not one of the main goals of the proposition. To see where the "normal propositions" come into play, let us jump to the middle of the proof where, as in V.51 (case [1]), AE is taken greater than half the *latus rectum*,[23] ΘE equals half the *latus rectum*, HΘ is twice AH, HB is set up perpendicular to AE, and EZ > Λ where Λ:HB::ΘH:ΘE. Apollonius continues as follows:

> We draw lines ZB, ZΓ.
> Then, if possible, first let line AZ be equal to BZ.
> We draw from point A line AK tangent to the section.
> Then line AK is perpendicular to axis AE, as is proven from Prop. 17 of Book I, because it is parallel to the ordinates falling on the axis.
> Therefore <ZAK is obtuse.

> So we draw from point A a perpendicular to AZ, namely AN: then it falls inside of the section, because it is not possible for any other line to fall between the tangent and the boundary of the section, as is proven in Prop. 32 of Book I. We draw from point B a tangent to the section, namely BΞ. Then the minimum line drawn between point B and the axis falls on the side of line BZ farther from point A, as we proved above.

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[23] The case in which AE is less than half the *latus rectum* is disposed of quickly through an appeal to V.49.

And [that minimum] forms a right angle with line BΞ,
as is proven in Prop. 27 of this Book.
So <ZBΞ is acute [emphasis added]*.

So if we make point Z center, and, with radius BZ, draw a circle, then [that circle] will cut line BΞ.

And NA will be outside of it [the circle];
for <ZBΞ is acute, and <NAZ is right.
So let the circle be circle BΞOA.
Then it cuts section AB: let it cut it at point O.
We join line OZ and draw line OΔ tangent to the section.

Then it [OΔ] falls outside of the circle, and the minimum line drawn between point O and the axis is farther from point A than is line OZ, as we proved [above]. *And it [the minimum line from O] forms a right angle with line OΔ, as is proven in Prop. 27 of this Book.*

So <ΔOZ is acute [emphasis added]*.
Therefore line OΔ cuts the circle.
But it [also] fell outside of it: that is absurd.
So line AZ is not equal to line ZB.

In the rest of the proof, where it is shown that AZ is not greater than ZB and "that, of the lines drawn from point Z, those closer to AZ are smaller than those farther [from it]," Apollonius relies on the same method used to eliminate the possibility that AZ equals ZB.

The crucial device in the method developed in V.64 is, undeniably, the circle BΞOA, and the crucial fact about BΞOA is that, besides points A and B, it cuts the section also at a third point, O, between A and B. Apollonius proves this by showing that AN, which is *inside* the section, is *outside* BΞOA, that BΞ, which is *outside* the section, is *inside* BΞOA, and then, by an implicit continuity principle of the sort one finds in *Elem.*,I.1, he concludes that BΞOA must cut the section between A and B. The contradiction comes, then, by showing that the tangent to the parabola at O must cut the circle and at the same time be outside the circle. That minimum lines are perpendicular to the tangents drawn at their ends is used twice (italicized above), once to show that <ZBΞ is acute and, therefore, that BΞ is inside BΞOA, and then, similarly, to show that <ΔOZ is acute and, therefore, also that OΔ is inside BΞOA. There is no doubt that V.27 is genuinely needed here, yet, what it is needed *for* is only to show the contradictory character of the circle BΞOA, rather than something essential about minimum lines as normals. In this respect, V.27 here, and the same could be said for V.72,73 as well, appears

to be, for Apollonius, *a mere lemma* "invoked for the purpose of establishing another [proposition] . . .," to quote Proclus.[24] One should note too that the proposition for which V.27 is being invoked happens to be about a minimum, AZ, *that is not also a normal*, a fact that, clearly, emphasizes further the ancillary character of V.27 in V.64.

So, what is the purpose of the part of Book V comprising V.27–33? We think that one can say with some confidence that, besides containing some useful "lemmas," this part of Book V is meant to show, within Apollonius's study of minimum and maximum lines, a study clearly more general, more precise, and more thorough than the inherited knowledge, the place of propositions known in some manner (except, perhaps, for the propositions in the group specifically about maximum lines) to Apollonius's contemporaries. Thus, in his opening letter, Apollonius writes, "You should realize that our predecessors and contemporaries paid (a little) attention only to the minimum [lines]: they proved thereby which straight lines are tangent to the sections and also the reverse, i.e. what properties are possessed by the tangents to the sections such that when those properties are possessed by the lines they *are* tangents. But as for us, we have proven those things in Book I without making use, in our proof of that, of the topic of minimum lines . . ." It is probable that the kind of reasoning known to Apollonius's contemporaries was as follows. One proves, first, by a proposition like V.29, which, as we have seen, demands none of the machinery developed elsewhere in the *Conica*, that if a line from the axis of a conic section to a point on the section is a minimum it is perpendicular to the tangent at the point where it cuts the section. Suppose, then, AE is a parabola having axis AB and *latus rectum*, p. Suppose further DE is tangent to AE at E, and EC is a minimum (see fig. 9). Then EC is perpendicular to DE. Also, sq.EB = rect.p,AB by I.11. But AB is half BD by I.35. So sq.EB = rect.p, half DB = rect.half p,DB. On the other hand, sq.EB = rect.CB,BD since <CED and <EBC are right. Therefore, rect.half p,DB = rect.BC,DB, from which it follows that

[24] Proclus, *In Eucl.*, p. 211 (the translation is taken from Morrow, *Proclus*, p. 165). We should add that in the same passage Proclus describes three methods by which one discovers lemmas, the third of these being "the reduction to impossibility" (the first two being "the method of analysis" and "the method of division (διαίρεσις)"). This is not to say that Apollonius *discovered* V.27 in the course of proving V.64, for V.27 was surely known previously (see below), but only to suggest that its place in V.64, precisely in such a "reduction to impossibility," adds to its lemma-like character.

BC is half p. Or conversely,[25] suppose one has proven something like V.8, then, with the same things being given, sq.EB = rect.half p,DB = rect.p, half DB. But sq.EB = rect.p,AB. So, rect.p,AB = rect.p, half DB, and, therefore, AB is half DB, which is, of course, the result in I.35. Indeed, one should recall that I.35, stating that in a parabola the segment of the diameter (extended) cut off between the tangent and the vertex equals the segment cut off between the vertex and the line drawn ordinatewise from the point of contact, is the *key* to the constructions of the tangent to a parabola in II.49,50.

Naturally, whether or not the reconstruction above was truly the way of Apollonius's predecessors we cannot know conclusively. However, it is clear that results like V.13 and V.27 *were* known, for we find them in the work of Archimedes[26] and Diocles. Let us consider the latter case, which refers to proposition 1 of Diocles' *On Burning Mirrors* (*Περὶ πυρίων*).[27] This case is particularly relevant because Diocles was probably a contemporary of Apollonius[28] and also because in *On Burning Mirrors*, prop. 1, we have an example in which the "normal" *as such* does play an important role. In proposition 1, Diocles demonstrates that rays of light entering a parabolic mirror (parallel to a diameter) will be reflected to a point on the axis of the parabola whose distance from the vertex is a fourth of the *latus rectum*. Paraphrased somewhat, the proof runs as follows:[29]

Let there be a parabola KBM [see fig. 10], with axis AZ, and let half the parameter of the squares on the ordinates be line BH. Let BE = BH, and let BE be bisected at point D. Let ΘA be tangent at

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[25] This follows a suggestion by J. P. Hogendijk reported by Toomer in *Apollonius* Conics *Books V to VII*, p. xxxix.

[26] Stated as if known in *On Floating Bodies*, II.5.

[27] Eutocius refers to the work by this name (see, for example, *Comm. in Archim. De Sphaera et Cyl.* ii., Archim. ed. Heiberg iii. 66). The Greek *text*, however, is lost; we have only the translation into Arabic to rely on. G. J. Toomer has provided us with an English translation and commentary in his *Diocles: On Burning Mirrors.*

[28] Heath places Diocles "a century or more later than Apollonius" (*History of Greek Mathematics*, II, 200). Toomer, however, disputes this, arguing 1) that Diocles was in personal contact with Zenodorus, whose flourishing in the early 2nd century B.C. can be established through his association with the Epicurean Philonides (mentioned also by Apollonius himself in the letter prefacing Book II), and 2) that, aside from Zenodorus, "Diocles refers only to mathematicians of Archimedes' generation or the preceding one, Conon, Pythion, and Dositheus" (Toomer, *Diocles: On Burning Mirrors*, pp. 1–2). Toomer's argument seems plausible.

[29] The translation is Toomer's in his *Diocles: On Burning Mirrors*, though we take the liberty to write, for example, "<O," instead of Toomer's "Ô," and "<O + <R" instead of Toomer's "O^R."

some point Θ, and ΘG be an ordinate to AZ. Then we know that AB = BG and that the line drawn from Θ perpendicular to ΘA meets AZ beyond E. So let us draw ZΘ perpendicular to ΘA, and join ΘD. Then GZ = BH = BE [This is the place we are using what in the *Conica* would be a combination of V.13 and V.27],[30] so, GZ = BE. From this and the fact that GB = AB (by *Conica* I.35), Diocles then shows easily enough that AD = DZ. Moreover, since triangle AΘZ is right, also DZ = ZΘ and, therefore, <O = <X. So, letting line ΘS pass through Θ parallel to AZ, we have <R = <O = <X, where <R is the angle ΘS makes with ΘZ. Therefore, also, <T = <P + <Q, from which Diocles can conclude that, ". . . when line SΘ meets line AΘ it is reflected to point D, forming equal angles, PQ and T, between itself and the tangent AΘ. Hence it has been shown that if one draws from any point on KBM a line tangent to the section, and draws the line connecting the point of tangency with point D, e.g. line ΘD, and draws line SΘ parallel to AZ, then in that case line SΘ is reflected to point D, *i.e.* the line passing through point Θ is reflected at equal angles from the tangent to the section. And all parallel lines from all points on KBM have the same property, so, since they make equal angles with the tangents, they go to point D."

In this proof, then, one has evidence of a proposition associated with Book V of the *Conica* used in a context where the "normal" really does play an important role. For to show that ΘD is, in fact, where the ray SΘ is reflected, Diocles must show that the angles T and DΘA are equal, and this he does by showing that the angles made by the incident and reflected rays, SΘ and ΘD, with the *perpendicular* to the tangent, ΘZ, are equal. Moreover, it is *as a perpendicular* to ΘA that ΘZ cuts off the segment GZ = BH = half *latus rectum*.[31]

[30] Toomer discusses this issue in *Diocles: On Burning Mirrors*, pp. 9, 150–151.

[31] We should point out that in what is called the "Bobbio Mathematical Fragment" (Heiberg, *Mathematici Graeci Minores*, pp. 87–88. See Toomer, *Diocles: On Burning Mirrors*, pp. 18–20 for an account of this fragment, and pp. 202–203 for the proof below) there is an alternate proof for Diocles prop. 1 that does not rely on the "subnormal property" and in which there is drawn no perpendicular to the tangent. The proof there is as follows. With KΘB a parabola, AD its axis, Y its *latus rectum*, and BD being a fourth of Y (see fig. 10a), we have "rect.Y,BG = sq. ΘG (property of the parabola). But Y = 4BD (construction). Therefore, 4rect.BD,BG = sq.ΘG. But BG = BA and ΘG = 2BC (similar triangles), therefore, rect.BD,BA = sq.BC. Therefore, <C is right. And since AC = CΘ,AD = DΘ, therefore, <DAΘ = <DΘA. [But <VΘS = <DAΘ, therefore, <DΘA = <VΘS.] But <AΘB = <SΘK, therefore, <DΘB = <KΘV [Toomer notes, as we should, the use here of "mixed" angles]*."

Now, this part of *On Burning Mirrors* at least,[32] is clearly a work of "applied" mathematics. It is an inquiry whose motivation comes from the world of physical phenomena and, perhaps, practical applications. Thus, the book opens, ". . . Pythion the Thrasian geometer wrote a letter to Conon in which he asked him how to find a mirror surface such that when it is placed facing the sun the rays reflected from it meet the circumference of a circle. And when Zenodorus the astronomer came down to Arcadia and was introduced to us, he asked us how to find a mirror surface such that when it is placed facing the sun the rays reflected from it meet a point and thus cause burning."[33] Indeed, following the proof of proposition 1, Diocles goes on to describe the burning mirror itself in material terms: "Hence, if AZ is kept stationary, and KBM revolved (about it) until it returns to its original position, and a concave surface of brass is constructed on the surface described by KBM, and placed facing the sun, so that the sun's rays meet the concave surface, they will be reflected to point D, since they are parallel to each other. And the more the (reflecting) surface is increased, the greater will be the number of rays reflected to point D."[34] Could such physical thinking, in which normals *do* have a place, have been anywhere near Apollonius's thoughts concerning maximum and minimum lines? We think not. For one, Apollonius never mentions any such applications or motivations in his prefatory letter to Book V or anywhere else in the *Conica*. And it does not seem probable that this is because he had

About this proof in the Bobbio Fragment, Knorr says "The most striking feature in Diocles' proof—its easy appeal to the property of the subnormal—is absent from the treatment in the fragment. But in a situation of this sort the later ancient editors typically insert the required lemmas, rather than eliminate the nonobvious assumptions through a general reworking of the proof. Further, an issue of no small concern about the fragment, the relation of rectilinear and curvilinear angles, is not addressed at all in Diocles' treatment here of the parabolic mirror. It thus seems more likely that the fragment is *not* based on the text of Diocles than that it is" (Knorr, "The Geometry of Burning-Mirrors in Antiquity," p. 58). Knorr suggests that the Byzantine Anthemius of Tralles was the true author of the fragment. Interestingly enough, the fragment itself attributes the work on burning mirrors, though almost certainly mistakenly (see Toomer, p. 20), to Apollonius.

[32] *On Burning Mirrors* appears to be a collection of theorems from various works loosely strung together rather than one integral work. Thus, Knorr writes, "In its extant form Diocles' book is actually a small anthology of geometric materials" ("The Geometry of Burning-Mirrors in Antiquity," p. 55). See also, *Toomer Diocles: On Burning Mirrors*, p. 3.

[33] Toomer, *Diocles: On Burning Mirrors*, p. 34.

[34] *Ibid.*, p. 48.

no knowledge of such work, for while he does not refer to Diocles directly he *does* refer to Conon in the letter prefacing Book IV and to Philonides (who is associated with Zenodorus) in the letter prefacing Book II;[35] certainly, he must have been aware of the kind of optical investigations represented by Euclid's *Optics*. One might be tempted to argue that Apollonius's discussion of the "points of application" [our *foci*] in Book III, props. 45–52 is motivated by optical studies, in particular, III.48, which demonstrates that, if a tangent be drawn to an ellipse or hyperbola, then ". . . the straight lines drawn from the point of contact to the points produced by the application [the *foci*] make equal angles with the tangent." But if this is true[36] and if, in addition, it is true that Book V is motivated by optical work, one would expect propositions such as these to be in Book V and stated in terms of "normals as minimum lines." Indeed, III.47 could be viewed as showing how a "normal" might be constructed from a point on a central conic. So, one could imagine a proposition in Book V stating that if Θ is a point on a parabola [see fig. 10] and a line SΘ is drawn through Θ parallel to the axis and another line ΘD is drawn from Θ to a point D on the axis such that the segment cut off between D and the vertex B is a fourth of the *latus rectum*, then the angle formed by the latter two lines is bisected by the minimum line through Θ. Needless to say, there is no such proposition in Book V. In fact, there is hardly a proposition in Book V where, for example, a line in a parabola is drawn parallel to the

[35] The stories about Archimedes and burning mirrors, which, if they were true, would make Apollonius's knowledge of such things likely, are hardly to be believed (see Dijksterhuis, *Archimedes*, pp. 28–29, and Knorr, "The Geometry of Burning Mirrors," pp. 53–55). As for the relationship between Diocles and Apollonius, see note 28 above.

[36] We tend to think that III.45–52 are *not* motivated by optics. First, there is the nagging question of Apollonius's not developing the analogous properties for the parabola. Zeuthen suggests that the case of the parabola was treated elsewhere, in the *Tangencies* (*Die Lehre von den Kegelschnitten im Altertum*, p. 371), while Toomer thinks that Apollonius "knew the treatment of Diocles and did not want to repeat it" (*Diocles: On Burning Mirrors*, p. 16). Yet, if optical investigations were behind III.45–52, one would expect to see *together* with these the case of the parabola which is so spectacular in the context of optics—and we do not think repeating Diocles' results would trouble Apollonius since he makes it quite clear in his opening letter that not everything in the *Conica* is original. Also, if the motivation and importance of these propositions were from optics, we would have trouble understanding their place in the books of "elements." What seems much more plausible is that III.45–52 are a natural continuation of propositions such as III.42 that clearly have nothing to do with optical studies.

axis, and where there is, say in V.68, nothing about it suggests that it is to be interpreted as a ray of light entering the parabola. The truth is that while Apollonius was no stranger to physical studies, as his work in astronomy attests,[37] from what texts we have we can say he kept "applied" and "pure" inquiries well apart. This, we think, could be said all the more regarding a text of such a fundamental character as the *Conica.*[38]

To summarize, having established in V.27–33 that minimum and maximum lines are perpendicular to tangents drawn at their ends and that the axis of a conic section cuts off a minimum or maximum line from perpendiculars drawn to tangents, Apollonius does not continue this line of thought, but that developed in the propositions preceding V.27–33. These seven propositions, central to the interpretation of minimum and maximum lines as "normals," are not used in the central theorems of the book, V.44–52, and in the few cases where they *are* used, V.64 being the leading example, they serve only as "lemmas" used to reach the contradiction in the *reductio ad absurdum* proofs; they are used nowhere in a way that modifies our understanding of what minimum and maximum lines are. Propositions V.27–33 are, therefore, clearly secondary to the general thrust of the Book. Moreover, previous knowledge of these and other propositions in Book V seen in Diocles' work in optics does not seem to have inspired Apollonius's work on maximum and minimum lines,

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[37] Ptolemy mentions Apollonius of Perga explicitly in the opening of Book XII of the *Syntaxis* (Heiberg, II, 450ff.) O. Neugebauer reconstructs what might have been elements of Apollonius's astronomy in his "Apollonius' Planetary Theory."

[38] A few additional remarks are in order here. (1) There is at least one instance where the terminology of Apollonius's astronomical work might have entered into his geometrical work. This was noted by Mugler: "On observe une transformation sémantique analogue dans l'expression κατὰ συζυγίαν, attestée avec un sens strictement astronomique chez Autolycus, mais appliquée par Apollonius à la géometrie des coniques" (*Dictionaire Historique de la Terminologie Géométrique des Grecs*, p. 14). (2) We find Neugebauer's account (in "The Astronomical Origin of the Theory of Conic Sections") of the origin of Menaechmus's procedure for cutting right cones in the making of sundials completely contrived, and so we cannot accept his suggestion that astronomical considerations are behind the whole theory of conic sections. (3) We hesitate to say that keeping "applied" and "pure" inquires separate is characteristic of all Hellenistic mathematicians because of the obvious counterexample of Archimedes, who, in his *Method*, shows how mechanical thinking can lead to pure mathematical results. Archimedes is careful, however, to point out the problematic nature of such a work and presents it as something unusual. And Archimedes himself is highly unusual among Hellenistic mathematicians.

though Apollonius was probably aware of such work. Briefly, one can rightly doubt that any significant things about "normals" gained from optical studies actually carried over to the *Conica.*

The Evolute

As said in the introduction to this chapter, the assumption that Book V is about normals to conic sections easily leads one who is already disposed to view the *Conica* in the light of modern developments to the subject of the *evolute.* The previous sections have, we think, presented enough evidence to undermine this assumption regarding Book V and the study of normals. Therefore, the *ease* with which one can pass from "normals as maxima and minima" to the "determination of the *evolute*" can no longer, *by itself,* support any claim that the *evolute* has a place in the *Conica.* But while this certainly weakens the conventional view that Book V contains "criteria [that] are tantamount to what we should describe as the equations of the evolutes to the conics,"[39] it does not utterly refute it. So let us reconsider the conventional view.

The propositions with which the *evolute* is associated are V.51,52. These give the conditions whereby no line, one line, or two lines from a given point below the axis will have a minimum line cut off between the axis and the conic section. The case in which only one such line may be drawn is the crucial case for the "determination of the *evolute.*" In V.51, as we saw above in a different context, there is a parabola ABΓ and its axis ΓZ (see fig. 7). EZ is drawn perpendicular to the axis such that ΓZ is greater than half the *latus rectum.* ZH is, then, set equal to half the *latus rectum,* ΓH is cut at a point Θ such that ΘH is twice ΘΓ, and ΘB is set up perpendicular to the axis. A line, K, is now found such that K:BΘ::ΘH:HZ. The condition, then, that there be only one line from E having a minimum line cut of from it between the axis and the section is that EZ equal K. The condition for the hyperbola and the ellipse in V.52 is somewhat more complicated. Here, ABΓ is a hyperbola or ellipse, EΓΔ its axis, and Δ its center (see fig. 11).[40] ZE is drawn

[39] Boyer, *A History of Mathematics,* p. 170.

[40] In fig. 11 we have included only the relevant elements of Apollonius's diagram.

perpendicular to the axis and such that EΓ is greater than half the *latus rectum.* Now, "ΔΓ is half the transverse diameter,

and ΓE > 1/2 *latus rectum.*

> Therefore (ΔΓ:ΓE) is less than the ratio of transverse diameter to *latus rectum.*
> So if we make (ΔH:HE) equal to the ratio of transverse diameter to *latus rectum*, point H falls between points Γ and E.
> We take two lines in continuous proportion between HΔ, ΔΓ, namely ΘΔ,ΔK [*i.e.*, HΔ:ΘΔ::ΘΔ:ΔK::ΔK:ΔΓ]*.
> Let KB be a perpendicular to the axis, and let the ratio of some line, Λ, to KB be equal to (ΔE:EH)·(HK:KΔ).[41]

For the hyperbola and the ellipse, then, the condition that there be only one line from Z having a minimum line cut of from it between the axis and the section is that EZ be equal to Λ.

How does one, now, go from here to the *evolute*? The key is seeing the case where ZE equals K for the parabola and where EZ equals Λ for the hyperbola and ellipse as *limiting cases* for the point of intersection of two "normals." Thus, Heath says, "It is clear that in case (b) [the case in which EZ equals Λ], O [our Z] is the point of intersection of two consecutive normals, or the center of the circle of curvature at P [our B]."[42] To make the connection, the reader should recall that the (*modern*) definition of an *envelope* for a family of lines (or curves) is that curve tangent to each of the lines in the family, or, as it is often put, the curve whose points are the points of intersection of "consecutive" lines in the family. In particular, the *evolute* is the *envelope* of the family of normals to a curve. If we now allow ourselves the free use of algebraic expressions and manipulations (which we shall do for the sake of the argument), we can "derive" the "equations" of the evolute to the conic sections easily enough. First, the parabola.[43] Taking ΓZ to be x and ZE to y, we shall find the equation relating x and y when (1) ZH = 1/2p = 2a (p is the

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[41] One may assume that in the original the last relation was "the ratio *compounded* of ΔE:EH and HK:KΔ."

[42] Heath, *Apollonius*, p. 177. This is the statement with regards to the hyperbola and ellipse; an identical statement is made regarding the parabola on p. 171 of Heath's text: "It is clear that in the second case where OM = y, O is the intersection of two consecutive normals, i.e. is the center of curvature at the point P_1." See also, Zeuthen, *Die Lehre*, pp. 289ff.

[43] In both "derivations," we are, more or less, following Heath (*Apollonius of Perga*, pp. 171, 177–8).

length of the *latus rectum*, a is a parameter), (2) $2\Gamma\Theta = \Theta H$, and (3) $y:B\Theta::\Theta H:HZ::\Theta H:p/2$. From (2) and (1), we have,

$$\Gamma\Theta = \tfrac{1}{3}(x - 2a) \text{ and } \Theta H = \tfrac{2}{3}(x - 2a)$$

But, from (3) and I.11, we also have,

$$y^2:B\Theta^2::\Theta H^2:\frac{p^2}{4}, \text{ or, } y^2:p\cdot\Gamma\Theta::\Theta H^2:\frac{p^2}{4}, \text{ or, in terms of a, } y^2:4a\cdot\Gamma\Theta::\Theta H^2:4a^2.$$

Therefore, multiplying both sides by $4a\cdot\Gamma\Theta$, we obtain,

$$ay^2 = \Theta H^2 \cdot \Gamma\Theta = (\tfrac{2}{3}(x - 2a))^2\ (\tfrac{1}{3}(x - 2a)) = \tfrac{4}{27}\ (x - 2a)^3, \text{ or,}$$
$$27ay^2 = 4\ (x - 2a)^3$$

Deriving the "equation" of the *evolute* to the hyperbola and to the ellipse takes a little bit longer, but it is, algebraically, no less straightforward. We shall find the equation of the *evolute* to the ellipse only. So, taking $\Delta E = x$ and $EZ = y$, we shall find the equation relating x and y when (1) $\Delta H:HE$::transverse diameter:*latus rectum*, (2) $EZ:BK = (\Delta E:EH)\cdot(HK:K\Delta)$, and (3) $\Delta H:\Delta\Theta::\Delta\Theta:\Delta K::\Delta K:\Gamma\Delta$. The algebra will be easier if we write ratios as fractions and proportions as equality of ratios. Thus, from (1) and I.21 (and where a and b are the lengths of the semi-major and semi-minor axis, respectively), we have,

$$\frac{\Delta H}{HE} = \frac{\text{transvers diam.}}{\text{latus rectum}} = \frac{a^2}{b^2} \quad (1')$$

Hence,

$$\frac{\Delta H}{\Delta H - x} = \frac{a^2}{b^2}, \text{ or, } \Delta H = \frac{a^2x}{a^2 - b^2} \quad (4).$$

From (3) and (4), then,

$$\frac{\Delta H}{a} = \frac{\Delta K^3}{a^3}, \text{ or, } \Delta K^3 = a^2\Delta H, \text{ so that,}$$

$$\Delta K^2 = \left(\frac{a^4x}{a^2 - b^2}\right)^{2/3} = a^2\left(\frac{ax}{a^2 - b^2}\right)^{2/3} \quad (5).$$

Now, by (1'), $\frac{\Delta E}{HE} = \frac{a^2 - b^2}{b^2}$, so that together with (3), one can write,

$$\frac{y}{BK} = \frac{\Delta E}{HE} \cdot \frac{HK}{K\Delta} = \frac{a^2 - b^2}{b^2} \cdot \frac{HK}{K\Delta} = \frac{a^2 - b^2}{b^2} \cdot \frac{\Delta K - \Delta H}{\Delta K}.$$

Substituting $\frac{\Delta K^3}{a^2}$ for ΔH, and using the relation $\frac{\Delta K^2}{a^2}+\frac{BK^2}{b^2} = 1$ (which can be "derived" from I.21), one obtains,

$$\frac{y}{BK} = \frac{a^2-b^2}{b^2} \cdot \frac{a^2-\Delta K^2}{a^2} = \frac{a^2-b^2}{b^2} \cdot \frac{BK^2}{b^2}, \text{ so that, } BK^3 = \frac{b^4y}{a^2-b^2}, \text{ or,}$$

$$BK^2 = \left(\frac{b^4y}{a^2-b^2}\right)^{2/3} = b^2\left(\frac{by}{a^2-b^2}\right)^{2/3} \quad (6)$$

Therefore, again using the relation $\frac{\Delta K^2}{a^2}+\frac{BK^2}{b^2} = 1$ as well as (5) and (6), we have,

$$\frac{a^2\left(\frac{ax}{a^2-b^2}\right)^{2/3}}{a^2} + \frac{b^2\left(\frac{by}{a^2-b^2}\right)^{2/3}}{b^2} = 1, \text{ or,}$$

$$\left(\frac{ax}{a^2-b^2}\right)^{2/3} + \left(\frac{by}{a^2-b^2}\right)^{2/3} = 1.$$

The equation for the *evolute* to the hyperbola can be "derived" in a way analogous to the above. In that case, one obtains:

$$\left(\frac{ax}{a^2+b^2}\right)^{2/3} - \left(\frac{by}{a^2+b^2}\right)^{2/3} = 1.$$

Problems in these "derivations" abound, of course. Obviously, there is the problem of blatant algebraic manipulations and interpretations which are hardly avoidable in such a derivation but that have no place in Greek mathematics. And, even accepting the algebraic manipulations, the objections against using equations to define curves, which we discussed in the chapter on the "elements," could equally apply here. Aside from these (though they are damning enough), we would like to discuss two more objections in more detail.

The first of these objections is that it is very hard to justify the emphasis in these derivations on what is only *one* of three cases in Apollonius's proposition. To see V.51,52 as propositions relating to the *evolute*, one must, indeed, see the case in which only one line from the given point can be drawn such that a minimum line is cut off between the axis and section as having priority over the others. One would expect, therefore, for this case to be treated *last*, or be

otherwise stressed in some more or less conspicuous way, as Apollonius *could* have done without violating the logic of the proof. As it is, Apollonius treats the three cases with *equal* care and emphasis; he is interested in three equally possible situations, ZE being greater than K, ZE being equal to K, ZE being less than K, in the case of V.51, for example. The segment K in V.51, as also Λ in V.52, is central to the proposition as *a fixed line segment functioning as a standard against which EZ may be compared* whenever EZ is drawn as prescribed. Thus, Apollonius's enunciation of V.51 begins, "But if the perpendicular we mentioned cuts off from the axis a segment greater than half the *latus rectum*, then I say that it is possible to generate a line such that when the perpendicular drawn is measured against it . . .," and then, case [1], case [2], and case [3]. The sequence is clear. First, a perpendicular (of *unspecified* length) is drawn from the axis with the sole condition that it cut off from the axis a segment greater than half the *latus rectum* (and that, only because the case in which the perpendicular cuts off from the axis a segment *not* greater than half the *latus rectum* has been treated already in V.49,50). Next, the *standard*, K or Λ, is generated, regardless of the length of the perpendicular—and here it is important to note that in the diagram for V.51 and V.52 K and Λ are *set apart* from the conic section, as a standard should be. Finally, the perpendicular is compared to the standard. With that, one can make two immediate observations. (1) The perpendicular is not set equal to the standard from the start, as it might be if this were the case Apollonius especially wanted to study. Rather, the perpendicular is first drawn and, having been drawn, *may be* equal to the standard as it *may be* less or greater. (2) There is nothing in the enunciation or the proof (see the section above, "The Importance of Normals," for the proof of V.51) to suggest that the reader is to imagine the perpendicular moving along the axis, and at every position having a length cut off equal to (the newly determined) K (or Λ), producing thus a curve.

The second objection concerns the very nature of the *evolute* as an *envelope* of lines. For while others agree that Apollonius did not actually investigate the *evolute*,[44] we think one can go further and say that

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[44] Toomer, rejecting his earlier position in "Apollonius" (in *DSB*), says "Zeuthen (followed by Heath) notes that Apollonius' construction allows one to find the locus of the centers of curvature of the different points on the section: this locus is a

Apollonius did not investigate *any* curve as an *envelope*, that this was not, for him, a legitimate way of defining a curve. Had he thought otherwise, one could point to *missed opportunities* to develop curves in this way. Indeed, at the risk of being circular, we would suggest as the first example of such a missed opportunity the *evolute* itself. For even one who fully accepts the algebraic derivations above must, nevertheless, grant that Apollonius fails to mention explicitly, or even hint, that *some other curve* may be obtained by means of propositions V.51,52; the best one can say is that that easily *follows* from these propositions (accepting, of course, the algebraic manipulations involved as legitimate). Thus, Heath's chapter on the subject[45] is not entitled, "Apollonius's Determination of the *Evolute*," but "Propositions *Leading Immediately* [emphasis added] to the Determination of the *Evolute*." The curve *itself*, the *evolute*, is just not in the text. However, to make the stronger argument, namely, that the absence of the evolute from the *Conica* is a result of the way such a curve is defined, let us look at another example of what we have called a "missed opportunity," one referring to the conic sections themselves.

The sequence of propositions, III.41–43, has been thought by Zeuthen and Heath, among others, to be the basis of a representation of the conic sections as an envelope.[46] III.41 concerns the parabola and states, "If three straight lines touching a parabola meet each other, they will be cut in the same ratio."[47] Thus, if ABC is a parabola (see fig. 12)[48] with tangents ADE, EFC, and DBF, then CF:FE::ED:DA::FB:BD. To see how this is connected to the parabola as an envelope, one notes that the first proportion CF:FE::ED:DA leads, *componendo*, to the proportion, CE:FE::EA:DA, and from here, *alternando*, to the proportion CE:EA::FE:DA. Since CE and EA are

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curve of higher order which is known in modern times as the 'evolute' of the conic . . . But there is no evidence that Apollonius was interested in investigating this locus" (*Apollonius* Conics *Books V to VII*, p. liii). See also Knorr who, coming closer to our own position, says "Extant materials, however, provide no direct support for the view that Apollonius developed his results on normals in this way, or that any ancient geometer defined the center of curvature for the points of a curve or the evolute of any curve" (*The Ancient Tradition of Geometric Problems*, p. 318).

[45] In *Apollonius of Perga*, pp. 168–179.

[46] See Zeuthen, *Die Lehre*, pp. 343–365, Heath, *Apollonius of Perga*, pp. cxxx–cxxxvii (this is really just an adaptation of Zeuthen's discussion), and Jones, *Pappus of Alexandria*, II, 522–523.

[47] As always, translations from Books I–III are taken from Taliaferro, *On Conic Sections Books I–III by Apollonius of Perga*.

[48] This and the next figure are only a partial rendering of Apollonius's diagrams.

fixed lines, the ratio CE:EA is a fixed ratio. Hence, with CE and EA given in position and in magnitude, the envelope of lines FD such that FE:DA::CE:EA is the parabola. III.42 concerns the hyperbola, ellipse, and circle, but not in a way analogous to III.41, except that it too concerns two tangents cut by a third (though the initial two tangents are not arbitrary). It states, "If in an hyperbola or ellipse or circumference of a circle or opposite sections straight lines are drawn from the vertex of the diameter parallel to an ordinate, and some other straight line at random is drawn tangent, it will cut off from them straight lines containing a rectangle equal to the fourth part of the figure to the same diameter." Thus, suppose BE is a hyperbola, ellipse, a circle, or one of the opposite sections and AB is a diameter (see fig. 13). Furthermore, suppose that from A and B, respectively, AC and BD are drawn parallel to an ordinate (and, therefore, in the cases of the ellipse, circle, and opposite sections, both AC and BD are *tangent* to the section; for the hyperbola, of course, only BD is tangent), and CED is a tangent to the section at E. Then,

rect.AC,BD = fourth part of figure to AB (*i.e.*, rect.AB, *latus rectum* to AB).

How this may be turned into a proposition about conic sections as envelopes is easy to see. For, given two parallel lines AC and BD, and a (fixed) rectangle O on AB, the envelope of lines CD cutting AC at C and BD at D and such that rect.AC,BD = O is either a hyperbola, ellipse, or circle (it is an ellipse or a hyperbola depending only on whether C and D are on the same side of AB or on opposite sides). Zeuthen[49] linked propositions III.41–43, persuasively, with two other works by Apollonius, *The Cutting Off of a Ratio* (Λόγου 'Αποτομή) and *The Cutting Off of an Area* (Χωρίου 'Αποτομή). About these, Pappus tells us that "The proposition of the two books of the *Cutting off of a Ratio* is a single one, albeit subdivided; and therefore we can write one proposition, as follows: through a given point to draw a straight line cutting off from two lines given in position (abscissas extending) to points given upon them, that have a ratio equal to a given one," and, similarly, "There are two books of the *Cutting off of an Area*, and again one problem in them, though subdivided. Hence they also have one proposition, in all other respects

[49] See Zeuthen, *Die Lehre*, pp. 344ff. and Heath, *Apollonius of Perga*, pp. cxxx–cxxxvii, and *History of Greek Mathematics*, II, 177–180.

similar to the one above, and differing in this respect alone, that one must make the two (abscissas) that are cut off, in the former case, have a given ratio, but in the latter case, enclose a given area. This is how it will be expressed: to draw through a given point a straight line cutting off from two lines given in position (abscissas extending) to points given on them that enclose an area equal to a given one."[50] From this, one can see how propositions III.41–42 (or, rather, their easily proven converse) together with the machinery in these two other works by Apollonius would allow one to draw a tangent to a conic section from a given point outside the section in a way different from that given in II.49. But while this connection between III.41 and *The Cutting off of a Ratio* and that between III.42 and *The Cutting off of an Area* is certainly conceptually justified and allows a certain insight into the unity of these two otherwise only loosely analogous propositions, we do not see it giving any weight to the claim that Apollonius has in mind the conic sections *as envelopes.* Indeed, while finding a tangent to a conic section through a given point can be done with *precisely* the material in the propositions and these two works, finding *a set of tangents* to a conic section given two other tangents such as in III.41–42 requires less. Suppose, as in III.41 (fig. 12), EA and EC are tangent to a parabola at points A and C, respectively. Choose at random a point D on EA. Then DA, EA, and EC are given lines. So to find the tangent DF from D, one only has to find, by *Elem.*VI.12 for example, a fourth proportional to EA, EC, and DA, since, as we noted above, III.41 implies that FE:DA::EC:EA. To find the family of lines determining the envelope, therefore, one only needs to consider points on EA, and this simple case requires no proposition from *The Cutting off of a Ratio.* Again, suppose, as in III.42 (fig. 13), AC and BD are tangents, say, to an ellipse at the ends of a diameter AB, namely, AC is tangent at A and BD at B. Also, suppose rectangle O is equal to the fourth part of the figure on AB. Choose at random a point C on AC. Then AC is a given line and O is a given figure, so that one can

[50] Hultsch, pp. 640, 642. The English translation is taken from Jones, *Pappus of Alexandria*, pp. 86, 88. These works of Apollonius are not extant in Greek, although there is an Arabic version of *The Cutting off of a Ratio*, and there is evidence that *The Cutting off of an Area* was known to Arabic mathematicians (see J. P. Hogendijk, "Arabic Traces of Lost Works of Apollonius," pp. 223ff.). *The Cutting off of a Ratio* was translated into Latin by Halley in 1706. Halley attempted also a reconstruction of *The Cutting off of an Area.*

apply a rectangle to AC equal to O, needing no more than *Elem.*,I.44. One now marks off a point D on line BD so that the line BD is equal to the other side of this rectangle. One has found, thus, the tangent CD from the point C. So, again, to find the family of lines determining the envelope, one needs only to consider points on line AC, and, besides III.42, one needs only propositions from Book I of the *Elements*. With all that, the main objection to this interpretation of III.41–43 is still that *nothing* is said in the text to suggest that in these propositions we are to consider a *collection* of tangents *defining* the conic sections, that somehow these propositions are different from all the others in Book III that give an account of the many possible relationships of ordinates, diameters, and tangents. Again, thinking of these as defining *loci* does not really help since that involves the same problems we discussed in our chapter above on the "elements." Before, leaving this topic of envelopes, we would like to give one more example of a "missed opportunity." This example, however, we shall take from Diocles to suggest that failing to look at curves as envelopes was not just a peculiarity of Apollonius alone.

Following the proposition in *On Burning Mirrors* in which Diocles shows that parallel rays of light entering a parabolic mirror focus at a point, Diocles goes on to discuss in the next two propositions what happens when parallel rays of light enter a semi-circular (or, rather, hemispherical) mirror. In proposition 2, he shows that such rays of light will be reflected into the first quarter of the diameter (drawn in the direction of the incoming rays of light) from the circumference of the circle. The proof is straightforward. We shall paraphrase. Let FMA be the circumference of a circle having center B (see fig. 14), let radius AB be drawn, and from points L and G on the circumference let lines LF and GS be drawn parallel to AB. Join BG and BL. Set arcs GZ and LM equal to arc LG (so that <BLM = <BLG = <BGL = <BGZ), <ZGX equal to <LGS, and <GLE equal to <FLM. Since, clearly, <FLB equals <EBL, also EL equals BE. But LE is greater than EA (*Elem.*,III.7), so BE is greater than EA. And BE is greater than EG, since LE is greater than LG (also by *Elem.*,III.7). Therefore, <EGB is greater than <EBG, which equals <SGB. Now, <LGB equals <BGZ and <LGS equals <ZGX, so <LGS is greater than <ZGE. But <ZGX equals <LGS. So, <BGX equals <BGS, which equals in turn <GBX, and, therefore, <BGX equals <GBX.[51] Hence, GX

[51] At this point, Diocles jumps to the conclusion.

equals BX, which is greater than half BG, or, what is the same, half AB. Therefore, since also BX and XA together equal AB, we have BX greater than XA. With that, Diocles concludes, "Then let AB be bisected at H. Hence it has been demonstrated that if we draw any number of parallel lines to FAM, the circumference of the circle, and they are reflected from circumference FAM so as to produce equal angles, then (the reflected lines) pass between points A and H, and no line among them passes between points B and H. The nearer one of the lines parallel to AB is to AB, the nearer to point H does its reflection (pass), and the farther one of them is (from AB), the nearer to point A does its reflection (pass)."[52] Proposition 3 sharpens this result by demonstrating, essentially, that the rays of light meeting an arc, GM, equal to a sixth of the circumference, will be reflected to a segment, XH, equal to a twelfth of the radius BA. "Thus," as Diocles says, "great heat will result in the space between points H and X."[53] In fact, the reflected rays of light focus along a curve, appropriately known as a "caustic," and this curve has a cusp precisely at H (see fig. 15), which accounts for the "great heat" there.[54] A caustic is a quite concrete example of an envelope, since it is the physical envelope of rays of reflected light, and can, thus, actually be seen; one need only look, for example, at the reflection of light off the sides of a cup into the bottom. Moreover, in the case of the circle the exact form of the caustic of reflection is the cardioid, which Knorr has related to the "cissoid" studied by mathematicians of Diocles' generation.[55] Thus it is striking that Diocles should not mention any such *curve* in connection with propositions 2 and 3. The reason, we think, is that the natural definition of the caustic suggested in this context, namely, *as an envelope of lines*, could not be considered by Diocles, even in this "applied setting," as a legitimate way of speaking about a curve. Otherwise, it would be an almost inexplicable lost opportunity.

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[52] Toomer's translation, *Diocles: On Burning Mirrors*, p. 56.

[53] *Ibid.*, p. 62.

[54] The caustic itself can be thought of as an evolute to another curve, the *orthotomic curve* relative to the circle. According to Bruce, Giblin and Gibson ("Caustics Through the Looking Glass," p. 48), the study of caustics dates back to Tschirnhaus in the late 17th century (see also Boyer, *A History of Mathematics*, p. 473).

[55] The nature of the "cissoid curves" is discussed in detail by Knorr in *The Ancient Tradition of Geometrical Problems*, pp. 246–263. Knorr and Toomer (*Diocles*, p. 24) doubt that the "cissoid," as a "cardioid," was the curve Diocles studied despite Diocles' name being associated with the "cissoid" since the 17th century.

In summary, let us remark that seeing the study of the evolute in the context of the *Conica* assumes that the notions of an evolute and envelope are as "ordinary" for Apollonius as they are for us, that their definitions involve no special conceptual difficulties for him, that Apollonius *must* have known such things. With these assumptions, one can then find the propositions in Apollonius's work *leading* to the evolute and envelope. But the "ordinariness" of evolutes and envelopes, for us, is built on our immersion in the kind of infinitesimal analysis born in the 17th century. It is, thus, no accident, that evolutes and envelopes were discussed *explicitly* only since the 17th century.[56] What the arguments above have shown is that, for one who does *not* have evolutes and envelopes in mind to begin with, there is little *in the text* of the *Conica* to *imply* such notions. For example, we saw that there is nothing in V.51,52 emphasizing, among the cases of lines cut off between a section and its axis, the case relevant to the evolute, nor is there any mention in V.51,52 of any *other* curve besides the conic section. III.41–43 may, through their association with *The Cutting Off of a Ratio* and *The Cutting Off of an Area*, be related to the problem of constructing a tangent from a point to a conic section, but they do not, by that, imply a *characterization* of the section itself in terms of its tangents, *i.e.*, as an envelope. Finally, not only is the notion of an envelope (and, therefore, of an evolute) absent from Apollonius's work, but also from that of his contemporary, Diocles. Indeed, had Diocles the notion of a curve as an envelope, it would be hard to understand why, in his discussion of the reflection of parallel rays of light in the arc of a circle, he should not mention the "burning *curve*," which is the *envelope* of the reflected rays, especially since it is likely that this particular curve, as a "regular" curve but not as an envelope, was known to mathematicians in Diocles' time.

Maximum and Minimum Lines as Longest and Shortest Lines

With the discussion of the *evolute*, we have come to the end of the critical part of this chapter. We must now ask, if not *normals*, what *are* maximum and minimum lines? To begin the consideration of

[56] See, for example, Boyer, *A History of Mathematics*, pp. 410–415 and Struik, *A Concise History of Mathematics*, p. 142.

this question, however, we want to say another word about normals. One argument *in favor* of viewing maximum and minimum lines as normals is that in places in Book V there are lines one can call "normal" without ambiguity but that one can call "minimum" only with qualification. For instance, here is the statement and proof[57] of V.24: "If a point is marked on any conic section whatever, then only one of the minimum lines drawn from the axis meets it.

> Let the section be, first, a parabola, AB, with axis BΓ [see fig. 16].
> We mark on the section point A.
> Then I say that only one of the minimum lines can be drawn from the axis to point A.
>
> [*Proof*:]
>
> For, if possible, let us draw two [minimum] lines, AΓ, AΔ. We draw from point A a perpendicular to BΓ, namely AE.
>
> Then line EΔ equals half the *latus rectum*, as is proven in Prop. 13 of this Book.
>
> And similarly also EΓ equals half the *latus rectum*: that is absurd.
> So only one of the *minima* can be drawn from the axis to point A.

Considering a *minimum line* as, literally, a *shortest line*, what exactly does it mean to draw from the axis a shortest line meeting a point on the section? Naively, one would surely say that the shortest line from the axis to point A is the line AE. Yet, AE though it must be *shorter* than either AΔ or AΓ, is not even a candidate in the proposition for shortest line. Had a point on the axis *been given*, as in V.13, referred to in the proof, then there would be little ambiguity here. For in proposition V.13, any line from Γ (see fig. 17) other than AΓ is truly longer than AΓ, and the proof of V.13 depends on this. In V.24, however, only a point *on the section* is given, so, if one insists that a *minimum* line is genuinely a *shortest* line, the temptation is great indeed to say that the shortest line from the axis to the point A on the section is the shortest line from A to the axis, *i.e.*, AE. For one whose thesis it is that Book V is about normals from the start, however, this proposition is not at all problematic since being a shortest or longest line is only secondary to being perpendicular to the curve. Indeed, AE *cannot* be a *normal* to parabola AB, for if it were, the tangent at A would be, by definition, perpendicular to AE, and,

[57] The statement actually covers both V.24 and V.25; V.24, presented here, demonstrates the proposition in the case of the parabola.

therefore, parallel to the axis BΓ, a possibility ruled out by *Conica*, I.24. But besides having to answer all the arguments against normals given thus far, one would also have to answer here why this proposition (and V.25,26, which are on the same theme) *precedes* the very propositions that clear up the whole issue, namely, V.27–33; up until V.24, one *could* say that "normals" are in the text *implicitly*, but if what makes V.24 unambiguous is that the line to be drawn to A is a "normal" rather than a "shortest line," then the subject of maximum and minimum lines as normals must be made *explicit* sometime *before* V.24, and it is not. So, the consistency one obtains by interpreting "maximum and minimum lines" as "normals" is only apparent, and one is left, therefore, with "maximum and minimum lines" being "longest and shortest lines," though ambiguously. To draw a minimum line from the axis to a point on a section means to find a point on the axis such that the line drawn from this point to the given point on the section is shorter than any other line drawn from the same point on the axis to a point on the section. Thus, AE, though it is the shortest length from A to the axis, is *not* a minimum line because from E there is a line to the section shorter than AE.

Although the expression above is, admittedly, a bit obscure, it does preserve the meaning of "minimum" as "shortest." This is important. Apollonius's starting point in Book V is, we think, genuinely "shortest and longest" lines in the ordinary sense. It is certainly his ending point, as one sees in the closing propositions of the book, V.64–77. Moreover, Apollonius never *defines* maximum and minimum lines, which means they must be, in some way, ordinary lines with ordinary characteristics for him. Further evidence is in the connection between Book V and propositions in Book III of Euclid's *Elements* which are unambiguously about shortest and longest lines in relation to circles. Indeed, Eutocius describes Apollonius's endeavor in Book V to be like that of Euclid in *Elem.*,III.8.[58] For some reason, Eutocius mentions *only Elem.*,III.8; he could also have mentioned the complementary proposition, *Elem.*,III.7, for this proposition too is clearly mirrored in Book V. These propositions from the *Elements* are as follows. *Elem.*,III.7: "If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle, that will be greatest on which the centre

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[58] *Eutocii commentaria in Conica*, Heiberg, II, 186.

is, the remainder of the same diameter will be least, and of the rest the nearer to the straight line through the centre is always greater than the more remote, and only two equal straight lines will fall from the point on the circle, one on each side of the least straight line." Thus, in the diagram (fig. 18), FA is the greatest of the lines through F, FD is the least, and FB is greater than FC, and FC than FG. In a similar vein *Elem.*,III.8 states, "If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest the nearer to that through the centre is always greater than the more remote, but of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least." So, DA passing through the center M is the greatest of the lines from D to the concave circumference AEFC (see fig. 19), DE is greater than DF, and DF is greater than DC, while DG, whose extension passes through M, is least of the lines from D to the convex circumference HLKG, DK is less than DL, and DL is less than DH. More than one proposition in the *Conica* Book V can be taken as analogous to *Elem.*,III.7: any of V.6,7,12, 16,17,19 would do. As an almost random choice, take V.16: "If a point is marked on the minor of the two axes of an ellipse such that the amount of the minor axis between it and the vertex of the section is equal to half the *latus rectum*, then, of the lines drawn from that point to the section, the greatest is the part of the minor axis which is equal to the *latus rectum*, and the smallest is the complement [*i.e.* remaining part] of the minor axis; and, of the other lines [so drawn], those of them closer to the maximum line are longer than those farther [from it]; and the excess of the square on it [the maximum] over the square on each of them is equal to the rectangle constructed on the line between the foot of the perpendicular from it and the end of the minor axis similar to the rectangle bounded by the minor axis and the excess of the *latus rectum* over it [the minor axis]." Thus, if AΓ is the minor axis of ellipse ABΓ with the center Π (see fig. 20), and Δ is taken on the axis so that ΓΔ equals half the *latus rectum*, then ΔΓ is the greatest of the lines drawn to the section through Δ, ΔA is the smallest, and the lines nearer to

ΑΓ are longer than those farther away. The proposition is extended in V.19, which tells us that if ΓΔ is *greater* than half the *latus rectum*, then, once again, ΓΔ will be the greatest of the lines drawn to the section through Δ and that lines closer to ΓΔ will be greater than those farther away. We remark also that the diagram in V.19 (see fig. 21) is particularly reminiscent of that in *Elem.*,III.7, especially when one notes that half the *latus rectum* in the ellipse corresponds, in the circle, to half the diameter,[59] so that the point H in V.19 marking the length from Γ equal to half the *latus rectum* corresponds to the center of the circle, E, in *Elem.*,III.7. The direct analogy to *Elem.*,III.8, which, as pointed out above, Eutocius seems to consider emblematic of the whole book, is V.18 or V.34, or, better, both taken together. In V.18, Apollonius continues to develop the series of propositions that began with V.16. Thus, Apollonius says in V.18, "Furthermore, we make the minor axis of the ellipse ΑΓ, the center N, and the line equal to half the *latus rectum* ΓΔ [see fig. 22]:

> then I say that ΓΔ is the greatest of the lines drawn from the point Δ to the section;
> and that the smallest of them is ΑΔ;
> and that, of the other lines which cut the section, those closer to ΓΔ are greater than those farther [from it],
> and as for those lines which fall outside [the section], those closer to ΑΔ are smaller than those farther [from it] . . .[60]

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[59] This can be seen most easily in the context of I.21, which, we remind the reader, tells us that "If in an hyperbola or ellipse *or in the circumference of a circle* [emphasis added] straight lines are dropped ordinatewise to the diameter, the squares on them will be to the areas contained by the straight lines cut off by them beginning from the ends of the transverse side of the figure, as the upright side of the figure is to the transverse . . ." In the case of the circle, the square on the ordinate *is equal* to the rectangle contained by the lines cut off by it beginning from the ends of the diameter, which can be inferred from *Elem.*,III.35 (or, alternatively, from the Porism to *Elem.*,VI.8 together with *Elem.*,III.31). Therefore, in the circle, the upright side (*latus rectum*) must be equal to the transverse side of the figure (which is any diameter of the circle). We should add that Apollonius never speaks about the *latus rectum* of a circle outright, but, as in I.21, he often *includes* the circle in propositions that refer to the *latus rectum.*

[60] As in V.16, the continuation states the exact degree to which the maximum is greater than the other lines, namely, ". . . that the square on ΓΔ exceeds the square on each of them [the other lines] by the amount of the rectangle constructed on the line between the point Γ and the foot of the perpendicular [from the end of the line] similar to the rectangle mentioned in the two preceding propositions [that contained by the minor axis and the excess of the *latus rectum* over the minor axis]*."

V.34 completes the picture by demonstrating that, "If a point is marked outside a conic section on a maximum or minimum line extended, then the smallest length intercepted between that point and the section (on lines drawn from that point on either side of the section [*sic*] but not extended to cut the section in more than one point) is the line which is the extension of the maximum or minimum line; and of the other lines, those closer to it are smaller than those farther [from it]."

In the Euclidean propositions, a point is taken on a diameter or its extension and lines from that point to the circle are compared; the greatest of such lines, or the least of such lines, is greatest or least in a way requiring no qualification. Now, while there is not enough evidence to make the strong claim that in Book V Apollonius intended to generalize these *particular* propositions from the *Elements*, the very close analogies just described make it likely that propositions *such as Elem.*,III.7,8 were on his mind in the writing of the book—at least the first part of the book. For, as pointed out above, in the propositions preceding V.24, a point is also first taken on the axis (or implied to be taken thus) and lines from this point to points on the section are compared. Here too there is no question as to what is meant by a minimum or maximum line: it is the shortest or longest of all lines drawn from the given point to the section. The conception of maximum and minimum lines in the beginning of Book V is, thus, identical to that in the Euclidean propositions. To this one might add the observation that in Book V Apollonius seems to give more weight to the ellipse, the closest analogy to the circle, than to the other conic sections. Of the seventy-seven propositions in Book V, fifty concern the ellipse, as opposed to twenty-two on the parabola and twenty-eight on the hyperbola.[61] The attention given to maximum lines partly accounts for this, but not entirely; it does not account, for example, for the last propositions in the book, V.73–77, relating only to the ellipse.

The strong analogy between Apollonius's work in *Conica*, Book V and Euclid's work in *Elements*, Book III, which we have seen thus far, is less apparent in Apollonius's development of the subject of

[61] These numbers are not of propositions *exclusively* relating to the ellipse, parabola, or hyperbola. Hence, for example, V.34, whose subject is "a conic section" with no further qualification, is counted as a proposition concerning the ellipse, as well as one concerning the parabola and the hyperbola.

shortest and longest lines following *Conica*, V.24 and in Euclid's further treatment of shortest and longest lines in *Elem.*,III.15. In *Elem.*,III.15, Euclid shows that "Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the center is always greater than the more remote." The sense in which this proposition continues *Elem.*,III.7,8 can be seen in its parallel formulation: some line X is greatest, and lines closer to X are greater than those farther away. To *Elem.*,III.15 also there is an analogy in the *Conica*. It is V.11: "The smallest of the lines drawn from the center of an ellipse to the boundary of the section is half the minor axis, and the greatest of them is half the major axis; and those [lines drawn from the center] closer to the longest line are greater than those farther [from it]. And the amount by which the square on each of those lines exceeds the square on the shortest line is equal to the rectangle constructed on the line between the foot of the perpendicular [from that line] and the center similar to the rectangle contained by the transverse diameter and the excess of it [the transverse diameter] over the *latus rectum*." But besides the obvious differences that come from working with an ellipse and not a circle and from Apollonius's greater precision in describing how the lines that are not the greatest or least differ from those that are, there is another subtle but, we think, telling difference between the two propositions. In Euclid's proposition, the lines of which it is said "the nearer to the center is always greater than the more remote," are *any* lines in the circle. *Elem.*,III.15 is a proposition comparing all chords in a circle. Proposition V.11, however, grows out of V.10, which shows how to find a minimum in an ellipse *from a point* on the major axis whose distance from the vertex is greater than half the *latus rectum*.[62] In

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[62] The exact statement of V.10 is "If a point is marked on the major axis of an ellipse, such that the distance between that point and the vertex of the section is longer than half the *latus rectum*, and the line between the vertex of the section and the point which was marked on the axis is cut at a point so that the ratio of [1] the line between the center of the section and the point at which the cut was made to [2] the line between that [latter] point and the first point which was marked is equal to the ratio of the transverse diameter to the *latus rectum*, and from the point at which the cut was made a perpendicular is drawn to the axis to meet the section, and from the point where it meets [the section] a line is drawn to the first point which was marked,

> then this [last] line is the smallest of the lines drawn from that point which was marked to the section, and, of the remaining lines [drawn from that point to the section], those of them closer to that line are smaller than those farther [from it];

V.11, then, one has the special case where the point on the axis is the center of the ellipse and so the lines compared in the ellipse are only those joining that point to points on the section rather than arbitrary chords. So, while V.11 is analogous to *Elem.*,III.15, the analogy is a restricted one. It may be, of course, that Apollonius thought that a *more direct* analogy to *Elem.*,III.15 such as "Of parallel straight lines in an ellipse that through the center is greatest, and of the rest the nearer to the center is always greater than the more remote" was unnecessary to state being an easy consequence of I.21 and *Elem.*,II.5. For *Elem.*,II.5 tells us that if a line AB (see fig. 23) is divided into equal segments at C and into unequal segments at another point D, then the square on CB (or AC) is equal to the rectangle contained by AD,DB together with the square on CD, from which it follows immediately that of all the rectangles contained by AD,DB the greatest is that where D is at C and of the rest those for which D is closer to C are greater than those for which D is farther away.[63] But, by I.21, the ratio of the squares on the ordinates to the rectangles contained by the segments they cut off from the diameter is always the same. Therefore, the squares on ordinates closer to the center must be greater than those farther away and, hence, also the ordinates closer to the center must be greater than those farther away. It seems to us more probable, however, that while Euclid, in *Elem.*,III,7,8,(14),15, is looking at longest and shortest segments *however* they might arise in relation to circles, in V.11 Apollonius is interested in longest and shortest lines *only in a particular geometric configuration*, namely, from a point on the axis to a point on the section or, put another way, *between the axis and the section.* If this is not clear in the first part of the book, it is when one arrives at V.24. What makes V.24 difficult is that while the given point is on the conic section, *the shortest line must be drawn as it*

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and the amount by which [each of] the squares on them exceeds the square on it is equal to the rectangle constructed on the line between the foot of [each of] the perpendiculars from them and the foot of the perpendicular from it [the minimum] which is similar to the rectangle contained by the transverse diameter and the excess of the transverse diameter over the *latus rectum*, when the transverse diameter corresponds to that line [the one between the feet of the perpendiculars].

[63] These conclusions are to be found in Pappus, *Coll.*,VII, Hultsch, pp. 694–696 where they are included among the lemmas to Apollonius's *The Cutting off of a Ratio* and *The Cutting off of an Area.*

was in V.6,7,8,9,10, or any of the other previous propositions. Thus, one can say that while Euclid considers shortest and longest lines in varying geometric circumstances, Apollonius considers one geometric arrangement arrived at in varying ways. The fixity of this geometric arrangement throughout the greater part of the book[64] is, perhaps, what causes one to think that a definition of a minimum line is somewhere needed in Book V; one wonders whether "minimum" is to be understood as a simple adjective describing "line," or, because of the specific geometric arrangement, whether "minimum line" is to be taken together as a single nominal phrase describing a peculiar geometrical entity.[65]

This concentration, which we have been discussing, on the geometrical configuration fixed in the opening propositions of Book V, rather than on the length of segments, pure and simple, discourages conceiving Apollonius as looking at the continuous changes in the length of lines drawn from a fixed point to a curve as the point of the line on the curve moves along the curve. Heath implies this kind of understanding of Apollonius's approach in Book V when he formulates his version of propositions such as V.7 as follows, "If any point O be taken on the axis of any conic such that $AO < \frac{1}{2} p_a$ [that is, such that the distance of O from the vertex, A, is less than half the *latus rectum*], then PA is the *minimum* strainght line from O to the curve, *and OP (if P is any other point on it) increases as P moves further and further from A* [emphasis added]."[66] Even Toomer does not hesitate writing that "The general problem solved in the book [Book V] may be stated as follows. For a given conic section and a given point P: (1) to construct all minimum and/or maximum straight lines from P to the conic; (2) *to determine how the distance between P and a*

[64] Propositions V.64–77, which are exceptional in this respect, will be treated below.

[65] Toomer's translation helps somewhat by fairly consistently using the word "minimum" where a line is shortest and in the geometric configuration of which we have been speaking, and using "shortest" otherwise. Thus, in V.12 one has "If a point is marked on one of the lines which has been proven to be a *minimum* [emphasis added]* of lines drawn from some point on the axis to one of the [three] sections, and lines are drawn from that [first] point to the section on one side, then the *shortest* [emphasis added]* of them is the portion of the *minimum* [emphasis added]* line adjoining the section, and those [lines] closer to it are shorter than those farther [from it]." The difference in the Arabic seems to us slight—though in such things we must defer to Toomer.

[66] Heath, *Apollonius of Perga*, p. 143.

variable point X on the conic changes as PX moves away from the position of minimum or maximum [emphasis added]."[67] Besides what we have said so far, other arguments make this approach to Book V dubious. Consider, for example, proposition V.75. This, as well as the other propositions in the group ending Book V, tempt one to think that the second part of Toomer's description is, indeed, appropriate, but a closer look shows that it is not. Here is the enunciation of V.75: "If a point is marked below the major of the two axes of an ellipse, and it is possible to draw from it to the section three lines such that the parts of them which the axis cuts off are *minima*, two of these lines being on that one of the two sides of the minor axis on which the point is, and one line being on the opposite side:

> then, of the lines drawn from that point to the segment of the section between the midmost of the three lines and that vertex of the section which is farther from the point,
> the greatest is that one of the three lines which is drawn to the side opposite to that in which the point is,
> and those of these lines closer to it are greater than those farther [from it];
> but as for the lines drawn from that point to the segment which is between the midmost of the three lines and that vertex of the section which is next to the point,
> the greatest of them is the line next to that vertex of the section which is on the side on which the point is,
> and those of these lines which are closer to it are greater than those which are farther [from it],
> and the greatest of these lines and [also] of the other lines which we mentioned previously is that one of the three lines which is drawn to the side opposite to the side on which the point is.

So, suppose ΑΒΓ is the ellipse (see fig. 24), ΑΓ its major axis, Ξ its center. Furthermore, suppose point Ε is taken below the major axis (and to one side of ΒΞ, the perpendicular to ΑΓ) and ΕΖ, ΕΔ, ΕΗ are such that the parts of these lines cut off between the axis and ellipse, ΘΖ, ΛΔ, ΚΗ, respectively, are *minimum* lines. Then, (1) among the lines in the segment of the ellipse, ΓΔ, the line ΕΖ is greatest and those lines nearer to ΕΖ are greater than those farther away, (2) among the lines in the segment, ΔΑ, ΕΗ is greatest, and, again, those lines nearer to ΕΗ are greater than those farther away, and

[67] Toomer, *Apollonius* Conics *Books V to VII*, pp. xxxviii–xxxix.

(3) of the two lines, EZ and EH, EH is the greater. In all this, the line EΔ appears only as a boundary between two segments of the ellipse. However, if we were to imagine lines from E to the ellipse *moving continuously* from A to Γ, one would expect Apollonius to point out, at least, that EΔ is the least line in the segment HZ. To use anachronistic terminology, though terminology fitting, perhaps, an investigation concerning the way "... the distance between [a given point] P and a variable point X on the conic changes...," one wants to say EH is a maximum, EZ a *local* maximum, and EΔ a *local* minimum. Needless to say, this way of speaking is not found here or anywhere else in Book V.

Let us review the argument. Apollonius's investigation into maximum and minimum lines seems to have its origins in propositions like *Elem.*,III.7,8. To this extent, the work in Book V begins with maximum and minimum lines as truly longest and shortest lines. But Apollonius is as much interested in a geometric situation in which a line has been drawn from a point on the axis of a curve to a point on the curve, or, a line cut off between the axis and the curve. The insistence on this geometrical situation typifies Book V and it is what often makes the characterization of maximum and minimum lines as longest and shortest lines ambiguous. Moreover, the weight given to the particular geometric configuration in which maximum and minimum lines are found is consistent with arguments against Apollonius's looking at maximum and minimum lines as maxima and minima in some continuous process, such as a point moving along a curve.

Neusis-like Constructions

We have shown, thus far, that minimum and maximum lines are not to be thought of as normals, nor are they to be thought of as longest and shortest lines without qualification. We seem to be no closer to understanding what *are* minimum and maximum lines. But, perhaps, we are asking the wrong question. Perhaps, we should be asking what is the Greek mathematical context for the study of minimum and maximum lines. So, let us try to categorize *the kind* of mathematical inquiry Book V represents. Of course, there is a sense in which Book V may not *lend* itself to categorization: for although, following Eutocius, we have compared the work in Book V with that

in *Elem.*,III.7,8, we have also seen that this comparison, at least, is limited, and it may be that the kind of mathematics represented by Book V of the *Conica*, in general, is *unique* to that book. It also may be that there is no *kind* of mathematics in this book, only a collection of theorems having a certain theme. Yet, Apollonius does seem to refer to the contents of Book V as containing a way of approaching conic sections, as well as containing definite truths about them. This can be discerned in his remark in the letter prefacing Book V that "... we have proven those things [regarding '... which straight lines are tangent to the sections and also the reverse, i.e. what properties are possessed by the tangents to the sections such that when those properties are possessed by lines they *are* tangents']* in Book 1 without making use, in our proof of that, of the topic of minimum lines ..." So, asking about the kind of mathematics in Book V seems to us at least a reasonable question. As an answer, we shall suggest that, if one were to divide Greek mathematics into families of approaches and procedures, then the work in Book V would belong to that family containing work related to *neusis*. Naturally, we are *not* claiming here that what Apollonius does in Book V *is neusis*, as we shall emphasize later, but that it is representative of a similar *kind* of mathematical thinking. To begin the discussion, let us look at the ways Apollonius refers to maximum and minimum lines.

In Book V, Apollonius describes maximum and minimum lines as (1) drawn "from the axis to the section," or "from a point on the axis," (2) "in the section" (we include here also places where Apollonius refers to maximum and minimum lines without any qualification at all), (3) drawn "from the section to the axis," or "from a point on the section," (4) "cut off" or "falling between the axis and the section," or, (5) just, "cut off by the axis." These different ways of referring to maximum and minimum lines do not correspond to different *cases* or *kinds* of such lines; as discussed in the last section, maximum or minimum lines are always to be conceived as longest or shortest lines drawn to the section *from a point on the axis.* What the different phrases represent, then, are different contexts in which maximum and minimum lines in this sense appear, different ways of arriving at the same geometric situation.

When one charts Apollonius's use of these phrases in the various propositions of Book V, one sees that they fall, more or less, into three blocks, each including twenty to twenty-five propositions (see table on p. 190). In the first block, the emphasis is on maximum

and minimum lines "from a point on the axis to the section." This block runs from V.4 to V.25. Next, we have maximum and minimum lines "in a section," and maximum and minimum lines "drawn from the section." This is from V.27 to V.52. Finally, we have maximum and minimum lines "cut off between the section and its axis," or, simply, "cut off by the axis," in the propositions V.53 to V.77. Of course, as one can see from the table below, these blocks are only *rough* groupings. For example, V.44 alone has Apollonius referring, one way or another, to minimum lines "from the axis," "in a section," "cut off between the axis and the section," and "from a point on the section."

As rough as the grouping of these blocks may be, one detects in them a progression in the geometrical conceptions they contain. In the first block, Apollonius explores the basic notion of maximum and minimum lines. It is in this block that the notion of a maximum or minimum line is least ambiguous, and, towards the beginning, not ambiguous at all. Moreover, it is here that the Euclidean analogy, discussed above, is strongest. The transitional propositions to the next block are, naturally, V.24–26. For, here, one learns that, as there is only one minimum or maximum line *from a point on the axis* to the section (that is, to one side of the section),[68] so there is only one minimum or maximum line *to a point on the section* (which proves, of course, that there is only one minimum or maximum line *from a point on the section*). In the course of this, also, one learns what it means to find a maximum or minimum line with respect to a given point *on the section.* Hence, one can now speak meaningfully of a segment being a maximum line or a minimum line *in a conic section*, that is, with one endpoint on the axis and one endpoint on the section. It is for this reason also, we think, that a maximum or minimum line being "from the section to the axis" often seems to go together with being "in a section," as in, for example, V.39, which states, "Maximum lines which are *drawn in an ellipse to the minor axis* [emphasis added]* cut each other in that part [of the ellipse in which they are drawn]." With the meaning of a minimum or maximum line in a section understood, Apollonius can speak of a maximum or minimum line cut off between the section and the axis. This may,

[68] Apollonius does not prove this outright, but it can easily be inferred after V.13–15 for minima and after V.22 for maxima.

Drawn from the axis, or, from a point on the axis.		In a section, or, unqualified.		Drawn from the section, or, from a point on the section.		Cut off or falling between the section and the axis.		Cut off by the axis.	
protasis	*ekthesis*	*protasis*	*ekthesis*	*protasis*	*ekthesis*	*protasis*	*ekthesis*	*protasis*	*ekthesis*
V.4–16, 19–20, 22–24	V.4–12, (13), 14–25	V.7	V.7, (13)			(23)			
V.32 44	V.30 47	V.27–31, 33–35, 37, 40–42, (46), 48, (51)	V.27–46, 48–50, 52	V.26 (38, 39) 44, 46 49, 51	V.26 44–46 49–52	V.44	V.44	V.47,51	V.51–52
		(V.62)	(V.62–3)	V.53–54	V.53–54	V.53–55, 58 64 (74)	V.53–55, 57–59, (60–61), 64–67, 74, (75–76)	(V.55) V.72, (73–74), 75, (76), 77	V.72–73, (74) (77)

on first sight, seem a separate issue, but it is not. For in the propositions where a maximum or minimum line is said to be cut off between the section and the axis, the maximum or minimum is a segment of a line drawn from some point neither on the axis nor on the section; one has, therefore, a line from a given point and such that the segment of it cut off between the section and its axis is a maximum or minimum line *in the section.* So, the way in which Apollonius refers to maximum and minimum lines in the last block clearly rests on the foundation developed in the second.

We do not want to claim that Apollonius designed the book, in some very conscious way, according to a plan based on these blocks; the blocks are simply not distinct enough and the specific content of the propositions contained in them is not uniform enough. On the other hand, we do think this division reveals something of the *direction* of Apollonius's thought. Indeed, there are places where, in one block, he seems to be *preparing* us for the next in a way that is not deductively necessary. Consider, for example, the very careful way Apollonius tells us how to draw the minimum line from the axis of a hyperbola to the hyperbola in V.9: "If a point is marked on the axis of a hyperbola such that the distance between it and the vertex of the section is greater than half the *latus rectum*, and the line between the point marked and the center is cut in two parts such that the ratio of one to the other is that of the transverse diameter to the *latus rectum*, the segment next to the center being the one corresponding to the [transverse] diameter, and there is drawn from the point at which that line was cut a perpendicular to the axis so as to meet the section, and the line between the point of its [the perpendicular's] meeting and the point which was [originally] marked is joined, then that joined line is the least of the lines drawn from the point marked to the section . . ."

Apollonius might have stated this proposition, as well as V.8 and V.10, in the form "If, from a point on the axis of a hyperbola, a line is drawn to a point on the hyperbola such that the perpendicular from this latter point to the axis cuts the line between the original point and the center in two parts . . .," and so on, that is, he might have stated the proposition in a way that would emphasize the motion *from the point on the axis to the section.* As it is, he states the proposition in the form of a detailed construction. This has obvious virtues. But it also has an interesting effect when one thinks of the procedure as actually carried out in time. First, a point is given, the

point on the axis; then a point on the section is found; then the line is joined. So, if one were to enter the scene precisely at the moment when the line was to be joined, one would find two points, of which one could not really say, "this is the first and this is the second." Indeed, Apollonius does not say join this point to that, but only *that* they are joined, and, immediately afterwards refers to ". . . that joined line . . .," as if the line has been dropped from above on two given points. In this way, then, it resembles the kind of procedure one expects in problems, not where one must draw a line having some property from a given point on a line to a curve, but where one must *fit a segment between a line and a curve*, or, *fit a segment in a curve.* Such a problem is *Elem.*,IV.1: "Into a given circle to fit (ἐναρμόσαι) a straight line equal to a given straight line which is not greater than the diameter of the circle."[69] The construction, paraphrased, is as follows. Suppose ABC is the circle and D is the given line (see fig. 25). Draw the diameter, BC, of the circle ABC (how to do this is implicit in *Elem.*,III.1). If D is equal to the diameter, then we are done. If the diameter is greater than D, then (from the point C on the diameter) cut off a segment CE equal to D and draw a circle EAF having center C and radius CE. Let CA be joined (where A is one of the two points of intersection of circles ABC and EAF). Then CA is equal to CE. So, CA is equal to D. "Therefore," Euclid concludes, "into the given circle ABC there has been fitted CA equal to the given straight line D." This is a very simple example of fitting a line into a curve. Nevertheless, one can see enough similarity between it and V.8–10 to suggest that the latter, together with V.24,25, might also be directed towards showing how a minimum line can be *fit* into a given conic section. The Euclidean proposition also suggests a setting against which one can think about the later

[69] Euclid tells us what it means "to fit a line in a circle" in *Elem.*,IV,def.7: "A straight line is said to be *fitted into a circle* (εἰς κύκλον ἐναρμόζεσθαι) when its extremities are on the circumference of the circle." Incidentally, the word ἐναρμόζεσθαι, deriving from the word ἁρμόζω (or, more radically, from ἀραρίσκω, to join or fasten), can have the musical sense "to be in tune," so one might say, whimsically, perhaps, that to fit a line given in magnitude into a circle is to set in the circle, as in a harp, a well-tuned chord (meaning, of course, that the string ought to be of the right length). In the next chapter, we shall return to the word ἁρμόζω in another guise when we look at the word ἐφαρμόζειν and its connection with Apollonius's understanding of the *equality* of conic sections.

propositions in Book V of the *Conica*, the ones grouped above in the third block. Let us now look at this setting.

Euclid's *Elem.*,IV.1 leads naturally to a problem in which one not only fits a line of a given length in a given circle but also requires the line, or its extension, to pass through a given point. This, of course, is an example of the kind of problem associated with *neusis*.[70] Apollonius wrote a work in two books on *neusis* (Νεύσεις). The work itself is lost, but we have a good idea of what it contained from Pappus's account of it and from fragments in Arabic sources.[71] About *neusis* itself, Pappus writes at the start of his description of Apollonius's work, "A line is said to make a neusis on a point (νεύειν λέγεται γραμμὴ ἐπὶ σημεῖον) if it passes through it when produced. Generally it is the same thing when (the line) is said to make a neusis on a given point (ἐπὶ δοθὲν νεύειν σημεῖον), or when some (point) is (said) to be given on it (τι ἐπ' αὐτῆς δοθὲν), or when it is (said) to be through a given point (διὰ δοθέντος ἐστὶν σημείου). They named these *Neuses* (νεύσεις) on the basis of one of these expressions, the problem being generally this: given two lines in position, to place (θεῖναι) between them a straight line given in magnitude, making a neusis on a given point. Among those that have varying assumptions in the details according to this definition, some were plane, some solid, some curvilinear. Choosing from among the plane ones those that are more useful for many things, they demonstrated the following problems: given in position a semicircle and a straight line at right angles to the base, or two semicircles having their bases in a straight line, to place a straight line given in magnitude between the two curves, making a neusis on the angle of the semicircle; and given a rhombus with one side extended, to fit (ἁρμόσαι) into the outside angle a straight line given in magnitude making a neusis on the opposite angle: and given in position a circle, to fit (in the circle) a straight line given in magnitude making a neusis on a given (point)."[72] As a an example, consider Pappus's *Coll.*,IV, prop. 31, which is an obvious variation on the case regarding the rhombus described above,

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[70] See Heath, *Euclid*, II,81.

[71] The fragments from Al-Sijzi's *Geometrical Annotations* are discussed and reproduced in Arabic with an English translation in Hogendijk's "Arabic Traces of Lost Works of Apollonius," pp. 194–206, 229–236.

[72] Hultsch, p. 670. The translation is A. Jones' (*Pappus of Alexandria: Book 7 of the* Collection, I, 112).

and which is preliminary to a solution by *neusis* of the problem of trisecting a given angle. Pappus writes, "Given a rectangular parallelogram ΑΒΓΔ and [the side] ΒΓ extended [see fig. 26],[73] let it be required to draw [the line] ΑΕ [so as] to make the straight line ΕΖ equal to a given [straight line]."[74] The synthesis, paraphrased, is as follows. Suppose ΑΒΓΔ is the given parallelogram and Μ is the given line. Set ΔΚ equal to Μ. Draw a hyperbola ΔΗΘ through Δ having asymptotes ΑΒ and ΒΓ (see *Conica*, II.4), and draw a circle about Δ passing through Κ and which cuts the hyperbola at point Η. Draw line ΗΖ parallel to ΔΓ and join ΑΖ (where Ζ is the intersection of ΗΖ and the extension of ΒΓ). Then ΕΖ is equal to Μ. For if ΔΖ is joined and ΗΛ is drawn parallel to ΚΑ, then rect.ΒΖ,ΖΗ will be equal to rect.ΒΓ,ΓΔ (*Conica*, II.12). As a result, ΓΔ:ΖΗ::ΖΒ:ΒΓ (::ΑΒ:(ΑΒ − ΓΕ)::ΑΒ:ΕΔ)::ΓΔ:ΔΕ. Hence, ΕΔ is equal to ΖΗ, so that, ΔΕΖΗ is a parallelogram and, therefore, ΕΖ is equal to ΔΗ, which is equal to ΔΚ, which is equal to Μ.

Neusis is also to be found in the description of the properties of curves, and, thus, found in a context including more than just straight lines and circles. We have in mind, naturally, the *conchoids* of Apollonius's near contemporary Nicomedes.[75] Pappus relates that, in order to solve the problem of duplicating the cube, Nicomedes generated a curve in the following way. Begin with a straight line ΑΒ and a line ΓΔΖ perpendicular to it (see fig. 27). On the latter choose a point Ε. Then, with Ε fixed, move the line ΓΔΖ so that Δ remains always on the line ΑΒ. The curve thus produced is a curve, therefore, with the following property (σύμπτωμα): "when any line drawn from Ε [falls] upon the curve, the [part] cut off (ἀπολαμβανομένην) between the straight line ΑΒ and the line ΛΓΜ is equal to the line ΓΔ."[76] As Knorr points out, "It is evident that the conchoids are merely an elaboration of the procedure of *neusis*, so that any problem solvable by means of the insertion of a given length between a given line and a second given curve, so as to pass through a given point, can be solved alternatively by means of the appropriate conchoid."[77]

The language in problems related to *neusis* is strikingly similar to

[73] Fig. 32 is the diagram accompanying the *synthesis* of the problem.
[74] Hultsch, p. 272, our translation.
[75] Assuming Knorr's placing Nicomedes' work in the last third of the 3rd century (*The Ancient Tradition of Geometric Problems*, p. 219).
[76] Hultsch, p. 242.
[77] Knorr, *op. cit.*, p. 221.

that in the propositions of Book V, especially those in which maximum and minimum lines are said to be cut off between the section and the axis. Consider, for example, V.55: "If a point is marked outside of one of the two halves of an ellipse into which the major of its two axes divides it, and a perpendicular is drawn from it [the point] to the axis so as not to fall on the center: *then there can be drawn from that point to the section a line such that the part of it cut off between the section and the major axis is one of the* minima [emphasis added]*, while [the line] cuts the other of the two halves of the major axis on which the perpendicular does not fall; and no other line can be drawn from that point cutting that half [of the axis] such that the part of it cut off is a minimum line." In the usual formulation of *neusis* problems, one must fit a line equal to a given line in a curve, or between two curves, and "verging towards (νεύουσα ἐπὶ . . .)" a given point. By Pappus's description, quoted above, other formulations are possible, including one in which the line is said to pass through a given point. This formulation is close to what one finds in the Apollonian proposition. One *could*, however, state the problem of V.55 in the more usual *neusis* formulation as follows: "Given a point outside of one of the two halves of an ellipse into which the major of its two axes divides it . . ., to fit a minimum line between the section and its axis and verging towards the given point . . ."

Of course, this is not unique to V.55. Even in V.62, which states that "It is possible for us to pass one of the *minima* through any point which is between one of the conic sections and its axis" one hears echoes of *neusis* when one considers the last problem mentioned by Pappus, ". . . to fit (in the circle) a straight line given in magnitude making a neusis on a given (point)." For, in the special case where the given point is *in* the circle, the point will be *on* the line fitted in the circle, so in that case the problem could be stated, "In a given circle, to pass a line of given length through any point which is in the circle." Needless to say, the proof of V.55, like that of many of the other propositions in this section, involves solid techniques such as those in the example above from Book IV of Pappus's *Collection.*[78]

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[78] To this extent, we can agree with Zeuthen's opinion (*Die Lehre*, pp. 284–309) that Book V presents an example of how the ancients deal with "solid problems," however, we do not see strong evidence that this is the *main goal* of Book V.

Further evidence for what we have been saying comes from a conspicuous feature of most of the diagrams in Book V, namely, that they consistently show only that part of a given section which is above the axis, as if the other side is completely unimportant. To see that this may have some real significance, let us again look at *Elem.*,III.7,8. At the end of the enunciations of both propositions, Euclid says "... only two equal straight lines will fall from the point on the circle, one on each side of the least straight line."[79] Euclid proves this in III.7 (the proof in III.8 is similar) as follows:

> I say also that from the point F [see fig. 18] only two equal straight lines will fall on the circle ABCD, one on each side of the least FD.
> For on the straight line EF, and at the point E on it, let the angle FEH be constructed equal to the angle GEF, and let FH be joined.
> Then, since GE is equal to EH,
> and EF is common,
> the two sides GE, EF are equal to the two sides HE, EF; and the angle GEF is equal to the angle HEF;
>
> therefore the base FG is equal to the base FH.
>
> I say again that another straight line equal to FG will not fall on the circle from the point F.
> For, if possible, let FK so fall.
> Then, since FK is equal to FG, and FH to FG,
>
> FK is also equal to FH,
>
> the nearer to the straight line through the centre being thus equal to the more remote: which is impossible.
> Therefore another straight line equal to GF will not fall from the point F upon the circle;
>
> therefore only one straight line will so fall.

In this part of III.7, then, Euclid takes it for granted that his proof in the previous part, which refers only to lines from F to the circumference *on one side* of the diameter, AD, is valid also for lines, such as FK and FH, *on the other side* of AD; it is a tacit recognition of the circle's symmetry. But what Euclid does prove here, namely, that a line from F to the circle can be equal only to one other line from F, that on the other side of AD, might also thus be taken for granted. He, however, does not take it for granted because III.7 is

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[79] This is the end of *Elem.*,III.7. The corresponding remark in *Elem.*,III.8 is "... only two equal straight lines will fall on the circle from the point, one on each side of the least."

directed towards all lines joining F and the circumference of the circle and its object is to find where these lines are lesser, greater, or equal. Moreover, although the first part of the proof might lead one to believe that Euclid looks at the lines first on one side of the diameter and then on the other, the last part of the enunciation shows that, in fact, he does not make this kind of separation. Thus, rather than saying "for every line that falls from the point to the circle on the one side of the least straight line there is one and only one such line equal on the other side," he says first, ". . . only two equal straight lines will fall from the point on the circle . . .," and only afterwards qualifies this and says, ". . . one on each side of the least straight line." Returning to Apollonius, one observes that this last statement in *Elem.*,III.7,8 we have been discussing *has no parallel statement in the analogous propositions in Book V.* In V.19, for example, Apollonius could easily enough have added, "I say also that from the point Δ only two equal straight lines will fall on the ellipse AB, one on each side of the maximum line ΓΔ." The proof, too, would have been similar to that of Euclid's in *Elem.*,III.7,8. For from E, say, draw a line EXT perpendicular to the axis (see fig. 28), and suppose it passes through the axis at X and the ellipse at T. Then, since EXT is perpendicular to the axis, EX is equal to XT. And ΔX is common. Therefore, ΔE is equal to ΔT. Now, suppose there is another line, ΔΨ, from Δ to the ellipse equal to ΔE. Then, ΔΨ is also equal to ΔT. But this is impossible, for ΔΨ is nearer to ΓΔ than is ΔT and, therefore, ΔΨ must be greater than ΔT.

Now let us look at the diagrams in Book V. In the other books of the *Conica*, the vast majority of diagrams suggest *whole* conic sections, and since, in the case of the ellipse or circle, one *can* depict the whole figure, most of the diagrams do. In Book V, however, one finds that the majority of the diagrams depict only the part of the conic section above an axis. For example, among the fifty propositions relating to the ellipse only *seven*, V.2,3,11,23,26,70,71, possess a diagram showing the whole ellipse. Of these, also, one can say that V.2,3 are really only theorems supplementing the "elements,"[80]

[80] V.1 states, "If there is a hyperbola or ellipse, and there is erected, at the end of one of its diameters, half the *latus rectum* to that diameter, at right angles, and a line is drawn from its [half the *latus rectum's*] end to the center of the section, and from [some] place on the section is drawn a line ordinate to the diameter: then that line [the ordinate] will be equal in square to twice the quadrilateral formed on half of the *latus rectum* in the way described in the example." V.2 demonstrates

so that, of the propositions whose subject matter properly belongs to Book V, only five concerning the ellipse possess a diagram showing the ellipse on both sides of the axis. The rest show only the half ellipse, the part above an axis (major or minor). The same can be said, too, for the majority of diagrams of the parabola and hyperbola. Let us call these diagrams, "hemidiagrams." For many of the propositions one might argue that the hemidiagrams are a mere convenience, that the proof given for the one half of the ellipse holds equally well for the other half. But we think there is more to it. For one, if mere convenience is the motive for these hemidiagrams, one wonders that there are *not* such diagrams to the same extent in *other* places in the *Conica*. For example, in I.21 (or, for that matter, in V.2), the whole ellipse is drawn, even though only one side of the diameter is referred to in the proof. Even in the propositions where it is not implausible to explain the hemidiagram as a matter of convenience, one is left with the sense that the other side of the figure would be, at least, helpful in seeing the fullness of the proposition. V.6, for example, states that ". . . if the same conditions as we mentioned hold [namely, that a point is marked on the axis such that its distance from the vertex of the section is equal to half the *latus rectum*]*, except that the section is an ellipse, and the axis its major axis, then the least of the lines drawn from that point is the one equal to half of the *latus rectum*, and the greatest of them the remainder of the axis. As for the other lines, those of them closer to the minimum line are less than those farther from it.

"And each of them [those lines] exceeds it [the minimum][81] by

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the proposition when, in an ellipse, the ordinate drawn passes through the center, and V.3, when the ordinate drawn passes through a point on the diameter on the other side of the center (*i.e.*, so that the center is between the point and the vertex from which the line equal to half the *latus rectum* is drawn). V.1 is used several times early in the book including the important proposition V.10 showing how to determine the minimum line from a point on the axis of an ellipse; V.3 is used in V.16–18, while V.2 is not used at all. The three propositions, to which Toomer refers as "lemmas" (*Apollonius* Conics *Books V to VII*, pp. xxxix–xl) form together a coherent sequence, so that one can explain the fact that V.2 is not used subsequently by seeing V.1–3 as a sequence of propositions belonging, in fact, to the books on "elements," but stated here because of the *immediate* use of two of them in the propositions of Book V. That these propositions follow easily from I.12,13 (cited in the proofs), that they are stated in terms of a general diameter rather than an axis, and that they are clearly reminiscent of the kind of propositions one finds in Books I–III supports this claim.

[81] About this obvious error, Toomer comments "Correct is 'And the square on

an amount equal to the rectangle constructed on the line between the foot of the perpendicular from it and the vertex of the section similar to the rectangle contained by the transverse diameter with the difference by which the transverse diameter exceeds the *latus rectum*, where the transverse diameter corresponds to the line between the foot of the perpendicular and the vertex of the section." Following this, in the specification, Apollonius says, "Let there be an ellipse, ΑΒΓ [see fig. 29], with major axis ΑΓ, and let ΓΔ be equal to half the latus rectum. And let us draw from point Δ to the section lines ΔΖ, ΔΕ, ΔΒ, ΔΗ." Of these lines, one, ΔΒ, is perpendicular to the axis, two, ΔΕ and ΔΖ, fall between the perpendicular and the part of the axis equal to half the *latus rectum*, and, one, ΔΗ, falls between the perpendicular and the remaining part of the axis. These appear to be, therefore, various *cases*, and, accordingly, Apollonius refers to each of ΔΖ, ΔΒ, and ΔΗ in his proof. However, ΔΕ, appearing in the specification, does not appear in the proof. Why not? Obviously, the proof for ΔΕ, if it were included, would be identical to that for ΔΖ. Also, it is probable that the reason ΔΕ is there in the first place is to stress that $ΔΑ > ΔΕ > ΔΖ > ΔΓ$ is true also when ΔΖ and ΔΕ are in the same quarter defined by the axis and the perpendicular ΔΒ.[82] These remarks concerning ΔΕ beg the question, why is there not, then, *even one* instance of a line drawn from Δ to the other side of the ellipse? If it is because the proof for such a line would be as that of any of the others lines already mentioned, why not, at least, *show* such a line, just as ΔΕ is shown? And if Apollonius is interested in stressing how the lines from Δ decrease in a series from ΔΑ to ΔΓ, why not stress how they begin to *increase* again as they go from ΔΓ back to ΔΑ on the other side? In other words, the same reasons justifying ΔΕ's being in the diagram but not in the proof

each of them exceeds the square on it.' Halley and Nix change the text; the error is probably the translator's" (Toomer, *ibid.*, p. 552).

[82] In ibn al-Haytham's text (H in Toomer's edition), one finds towards the end of the proof, "And the rectangle constructed on ΓΑ is greater than that on ΓΜ, and that on ΓΜ [greater] than that on ΓΔ, [*and that on ΓΔ greater than that on ΓΛ greater than that on ΓΚ* [emphasis added]*.] These rectangles are those mentioned in the second part of the enunciation: they are the excesses of the squares on the lines ΔΑ, ΔΗ, ΔΒ, ΔΕ, and ΔΖ over the square on ΔΓ. Since the rectangles are, by the proof, all similar, the statement implies that $ΔΑ > ΔΗ > ΔΒ > ΔΕ > ΔΖ > ΔΓ$. From the addition (italicized above) in al-Haytham's text, which contains the only reference (by referring to ΓΛ) to ΔΕ after the specification, one gathers that al-Haytham also saw that the only thing important to note about ΔΕ is its place in the sequence ΔΑ, ΔΗ, ΔΒ, ΔΕ, ΔΖ, ΔΓ.

are, on the face of it, even better reasons for a point on the other side of the axis being in the diagram. Thus, the argument that Apollonius omits one half of the ellipse because the proof for the included side works just as well for the omitted side does not hold, since, if it did, it would hold, *a fortiori*, for ΔE as well, and ΔE is *not* omitted.

In the case of V.6, the omission of the second side of the ellipse does affect the *proof* of the proposition; the lines Apollonius chose in the specification of the proposition are sufficient to prove exactly what was stated in the enunciation. This, however, is less clear in V.20, the first part of which states, "If a point is marked on the minor of the two axes of an ellipse, such that the line between that point and the vertex of the section is smaller than half the *latus rectum* but greater than half the [transverse] diameter, and the line between the vertex of the section and its center is divided at a point such that the ratio of the line between the center and that point at which the line was divided to the line between that point and the first point which was marked is equal to the ratio of the transverse diameter to the *latus rectum*, and there is drawn from this last point which was marked a perpendicular to the axis to meet the section, and a line is drawn from the point where it [the perpendicular] reaches [the section] to the first point which was marked,

> *then the greatest of the lines drawn to the section from that first point which was marked is the line which was joined,*
>
> *and, of the other lines, those closer to it are greater than those farther [from it]* . . . [emphasis added]*"

Taken *literally*, the conclusion (italicized) is not true. For, with ABΓ being the ellipse (see fig. 30),[83] AΓ the minor axis, Δ the first point chosen, and ZM the perpendicular described above, extend ZM to ζ on the section on other side of AΓ. Clearly, then, Δζ also satisfies the conditions for being the "greatest of the lines drawn to the section from that first point," Δ. Moreover, lines close to Δζ will not be smaller than, say, ΔΓ though they are farther away from ΔZ than is ΔΓ. An even more pointed example is V.21. It says, "If a point is marked on the maximum line mentioned in the preceding proposition in the ellipse such that the distance between it and that end

[83] This and the next diagram are not Apollonius's diagrams, though we have used the same lettering for corresponding points, where there are such.

of the maximum line which lies on the section is greater than the maximum line, then the greatest of the lines drawn from that point [to the section] in one part of the section is the line of which the maximum line is a part; and as for the lines on either side of it those of them nearer to that line are greater than those farther [from it]." Here, Apollonius does pay attention to the two sides of the ellipse in saying ". . . the greatest of the lines drawn from that point [to the section] *in one part of the section* [emphasis added] . . .," but he seems to be referring only to the "greatest line." Interestingly enough, *for this line*, he needs no qualification. However, for the second part, namely, "as for the lines on either side of it [the maximum]* those of them nearer to that line are greater than those farther [from it]," he does need some qualification *if* one considers the other half of the ellipse. For let ABΓ be the ellipse (see fig. 31), AΓ its minor axis, and ΔB a maximum line drawn from Δ. Furthermore, let Δβ be the corresponding maximum from Δ to the part of the ellipse on the other side of AΓ (obtained in the same way Δζ was found above in connection with V.20). Suppose E is chosen on ΔB such that BE is greater than ΔB. Then, for points E sufficiently close to Δ it will *not* be true, if one takes the *whole* ellipse into account, that lines closer to EB will be greater than those further away. In particular, for points sufficiently close to Δ, EΓ will not be greater than Eβ. Let us find the precise *diorism* in the case when <BΔβ is an obtuse angle. In this case, clearly, also <ΓΔE is obtuse, and one can choose E close enough to Δ so that also <ΔEβ is obtuse. Draw ΓP and βΩ perpendicular to BE produced. Then, by *Elem.*,II.12, the square on ΓE is greater than the squares on ΓΔ and ΔE by twice the rectangle contained by EΔ and ΔP. Similarly, the square βΔ is greater than the squares on βE and ΔE by twice the rectangle contained by ΔE and EΩ. Hence, the excess of the square on βΔ over the square on ΓΔ is greater than the excess of the square on βE over the square on ΓE by twice the square on ΔE and twice the rectangles contained by ΩE and ΔE and by PΔ and ΔE, or, taking ΔE as the common height, by the rectangle contained by ΔE and twice PΩ. Therefore, βE will exceed ΓE when <ΔEβ is obtuse and the rectangle contained by ΔE and twice PΩ is less than the excess of the square on βΔ over the square on ΓΔ. The point is that not only is Apollonius's statement in V.21 that ". . . of the other lines [other than EB]*, those closer to EB are greater than those farther away [from it]" strictly true only for the part of the ellipse above the minor axis, but also

considering the whole ellipse leads one to a *diorism* that is not completely trivial. Thus, in this case, it is impossible to explain the omission of the part of the ellipse below the minor axis in terms of mere convenience.

In V.21, we saw that Apollonius does refer to the other side of the ellipse. He does so in other places as well in Book V. For example, in V.49, we have, "If there is a conic section, and there is drawn from its axis a perpendicular to the axis such that that perpendicular cuts off from the axis on the side adjacent to the vertex of the section a line no greater than half the *latus rectum*, and a point is marked on that perpendicular and any line is drawn from it to *the other part of the section* [emphasis added]* between the perpendicular and the vertex of the section: then the minimum line drawn from the extremity of that line is not part of that line . . ." Or, again, in V.38, "If there are drawn in one of the conic sections two minimum lines ending at the axis, then, when they are extended in a straight line, they will meet in *the other part of the section* [emphasis added]*." Of course, he does not say explicitly that the two minimum lines are to be taken in one side of the section, but, *here*, one can say that this *is clearly* implied by the last words of the conclusion (italicized above). So, one cannot accuse Apollonius of being unaware of the other side of the ellipse. On the contrary, against the alternative that propositions in Book V are flawed by incompleteness or imprecision, what the hemidiagrams and the other arguments above suggest, we believe, is that Apollonius was quite deliberately *focusing* his attention on only one side of the ellipse, or, rather, on what can be placed between the axis and the ellipse. This, of course, is completely consistent with what we have been saying about *neusis*, for the field of action in *neusis* is precisely the space between the curve and the line; drawing the other side of the curve, where there is another side, would be a mere distraction in such an investigation.

So, is Book V a book about *neusis*? One must answer that it is not. On the other hand, it appears to derive from the same *kind* of thinking as that guiding *neusis*. Evidence for such a *kind* of thinking comes from what may be the earliest use of *neusis* in the work of Hippocrates of Chios, as it has come down to us through Simplicius. In the course of showing how one squares a *lune* having the outer circumference (EKBH in fig. 32) less than a semicircle, Hippocrates says, ". . . let the straight line EZ be placed between this [ΓΔ] and

the circumference [of semicircle AEB] verging towards B so that the square on it is one-and-a-half times the square on one of the radii (ἡ δὲ ἐφ' ᾗ ΕΖ κείσθω ταύτης μεταξὺ καὶ τῆς περιφερείας ἐπὶ τὸ Β νεύουσα τῶν ἐκ τοῦ κέντρου ἡμιολία οὖσα δυνάμει).[84] This shows that in the early tradition of *neusis*, at least, one did not always say *explicitly* that the fitted line be "*equal* to a given line." But, of course, this is not the same as saying *neusis* involved placing a line having *any given property* between two curves and verging towards a given point. Moreover, from what we can tell from Pappus, Apollonius's own work in *neusis* concerned itself only with placing lines *equal* to a given line either between two straight lines, or in a circle, or between two circles, or between a straight line and a circle. Still, the remarkably similar formulations in classical *neusis* problems and in the greater part of Book V justifies our calling the propositions in this book *neusis-like.* Seeing the book in the light of *neusis* takes much of the obscurity out of propositions that speak of drawing a line from a point to a section so that the line cut off between the section and its axis will be a minimum or a maximum without resorting to talk about normals. In other words, if Book V is guided by a *neusis*-type way of thinking, as we think it is, then the ahistorical reading involving normals and evolutes becomes even more offensive than it already is on the basis of the concrete textual basis alone.

Conclusion

The first part of this chapter asked the question whether it is appropriate to speak of normals in connection with Book V of the *Conica.* It is certainly understandable why writers in the past have been tempted to do so. Besides the immense importance of normals in mathematics since at least the 17th century, thinking of Book V in terms of normals clarifies, superficially (though *only* superficially), certain difficulties in the book. For if one can replace the words "minimum" or "maximum" with the word "normal," then, for example, one can say that the expression "to draw a line from a point to a section so that the line cut off between the section and its axis will

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[84] Simplicius, *In Physica*, ed. Diels, p. 64. English translation is by I. Thomas, I, 243–245.

be a minimum or a maximum" becomes simply "to draw from a given point a normal to the section." But, while a *normal* is a geometric notion, it is one whose significance is perspicuous only in connection with the study of curved lines *in general*, and this is built on the sort of analytic understanding of curves possible *only since* the 17th century. More importantly, however, the propositions before the "normal propositions," V.27–33, do not lead to the latter, and the propositions following V.27–33 do not flow from them. The whole issue of normals, that is, of lines perpendicular to tangents, seems, on the whole, a mere side issue in Book V. Without normals as a unifying theme, however, one is left with some difficulty saying what the book is about. The beginning and the end of the book are truly about shortest and longest lines from various points inside and outside of conic sections, and seem to take as their model propositions from the third book of the *Elements*. But in the middle and main part of the book this cannot be said unambiguously. The concentration there and throughout on the region *between* the section and its axis, and the use of language very close to that of *neusis*, suggests that the general effort in Book V is one that can be compared to that of *neusis*. It is representative of a *kind* of mathematics in which one speaks of fitting lines in figures or other figures in figures such as we saw in *Elem.*,IV.1. This puts the inquiry of Book V in a historically more sensible context, while suggesting, at the same time, that *neusis* itself might have belonged to a larger family of approaches and procedures than has previously been thought. In the next chapter, we shall consider the sixth book of the *Conica* in which the *kind* of inquiry is less in question than the *weight* such an inquiry receives both in Apollonius's and our estimation.

Figures

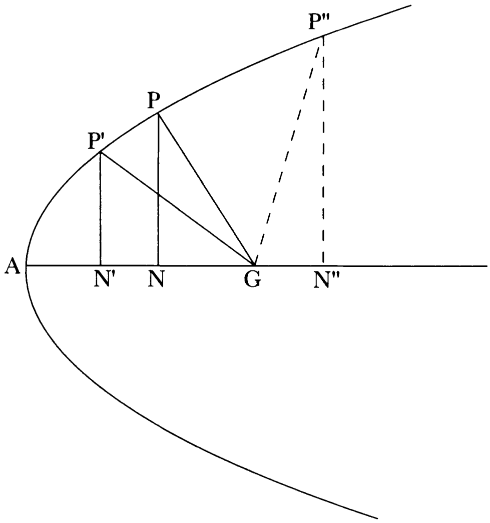
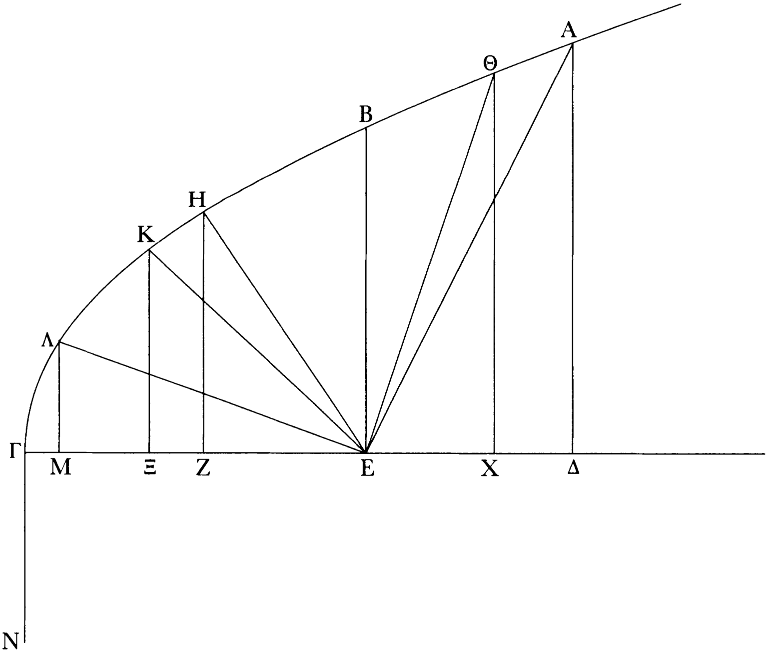


Fig. 1



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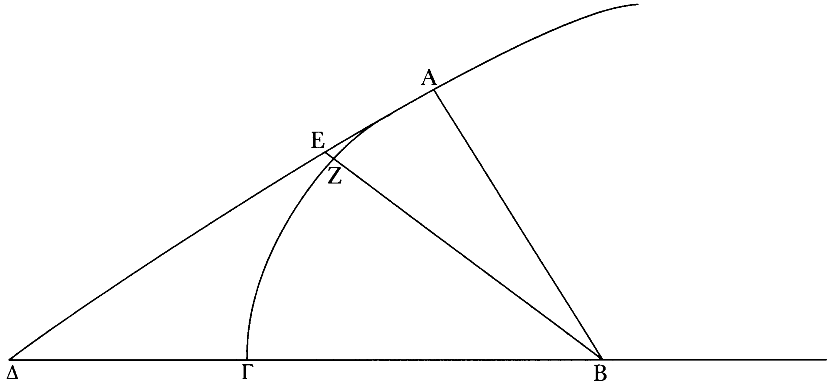

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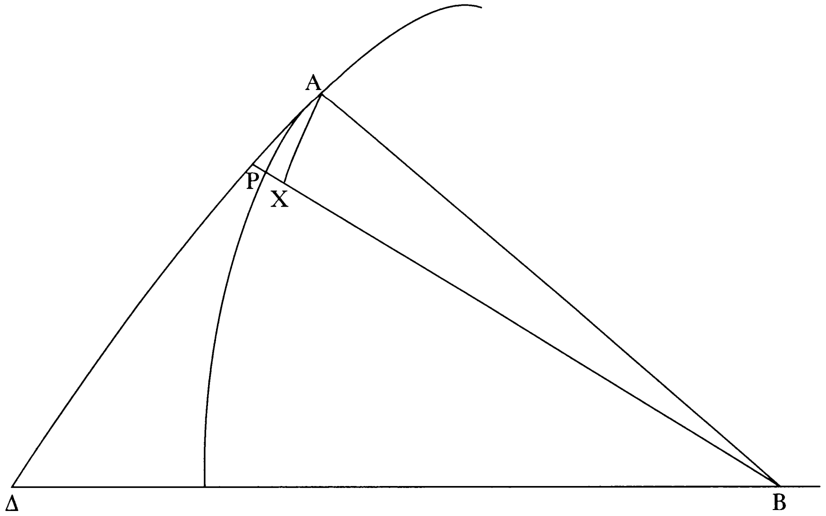

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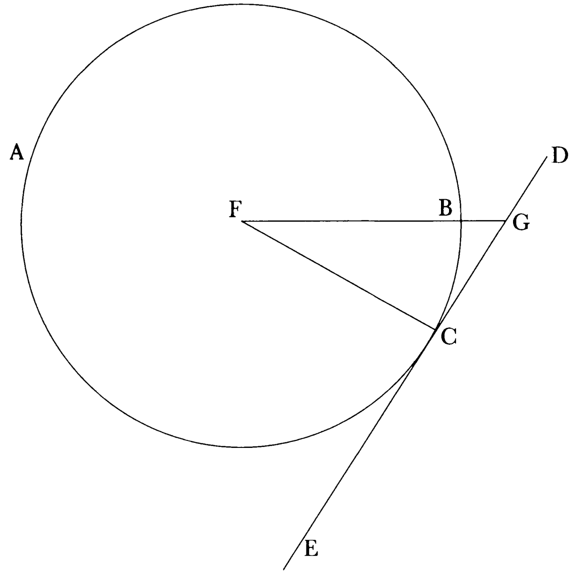

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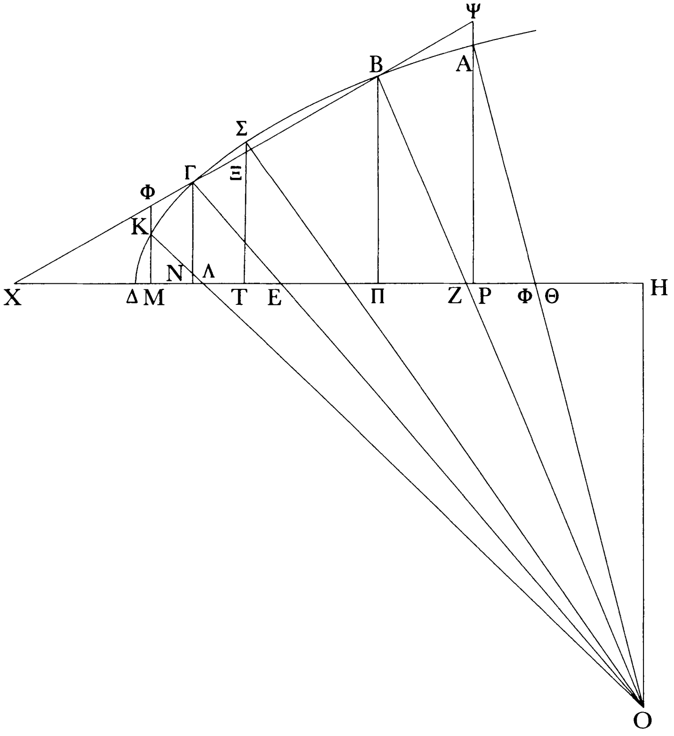

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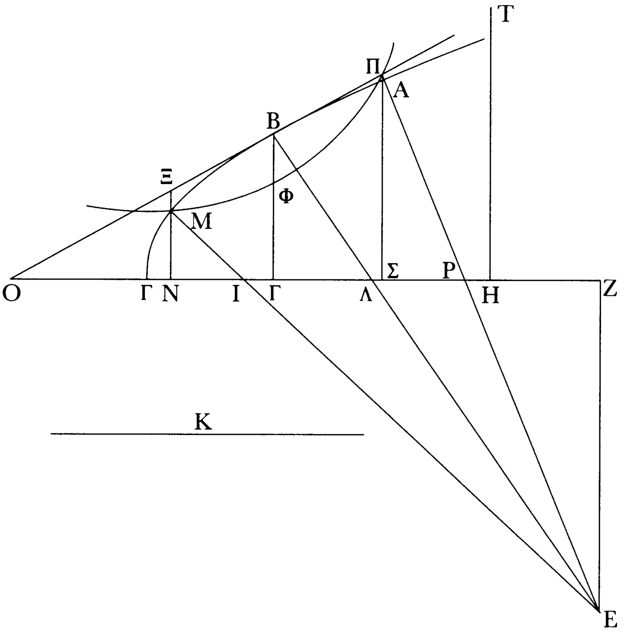

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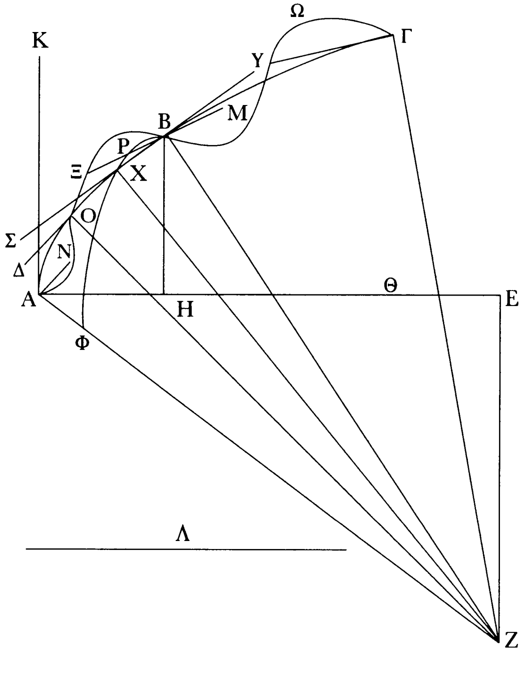

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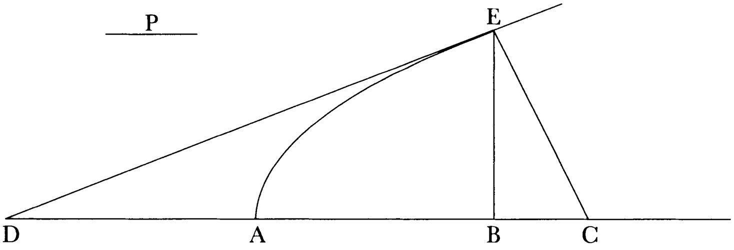

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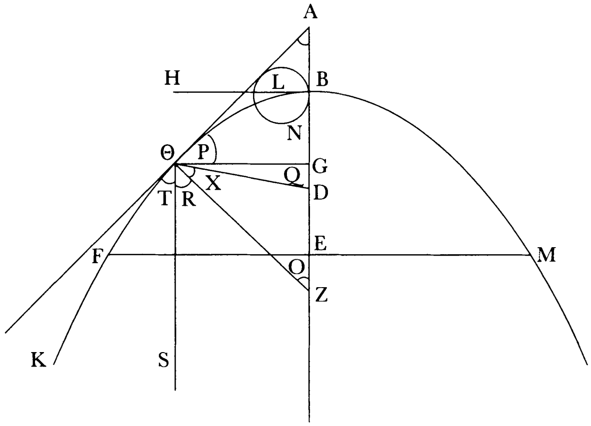

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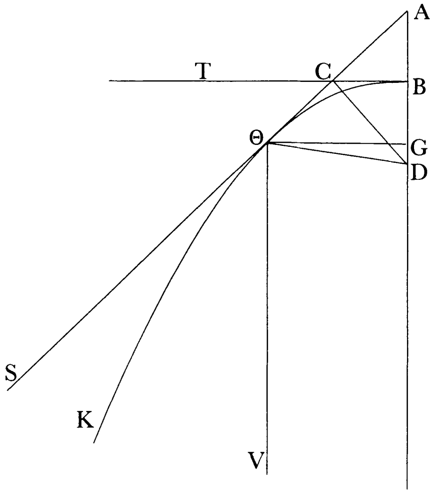
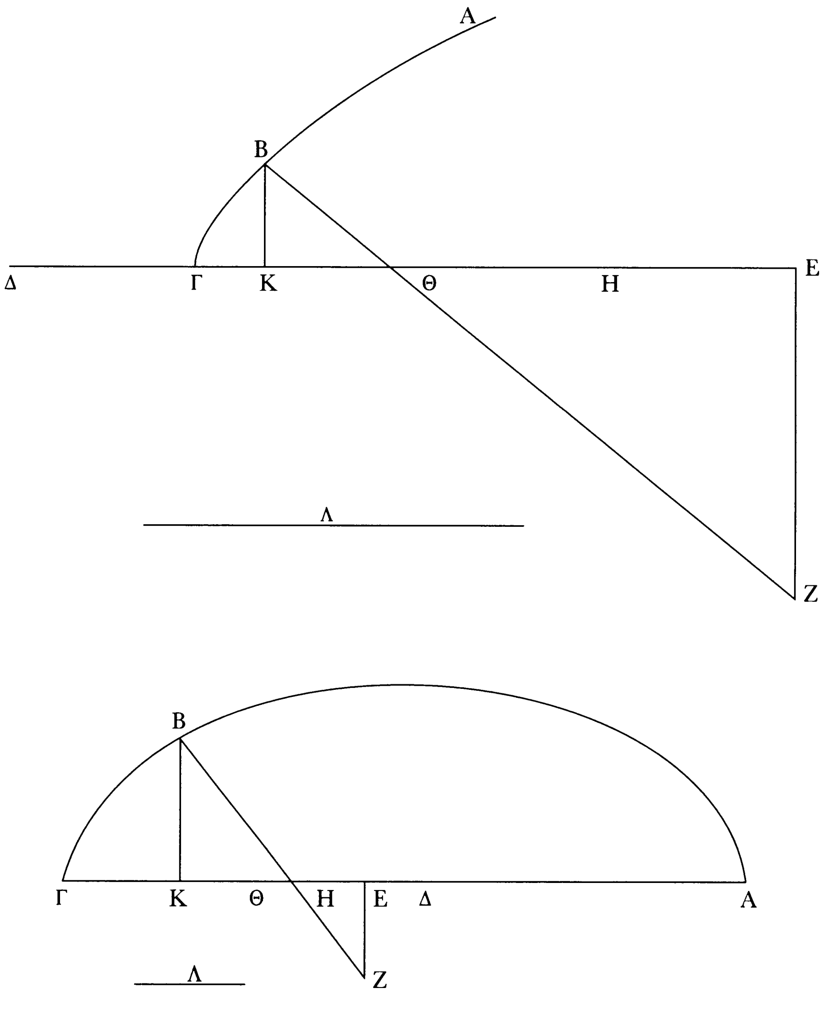


Fig. 10a



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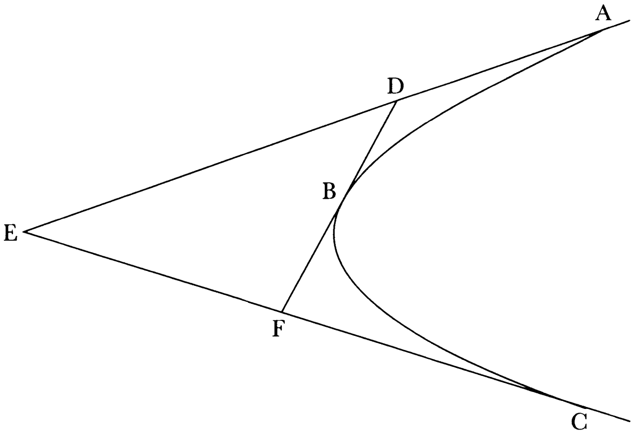

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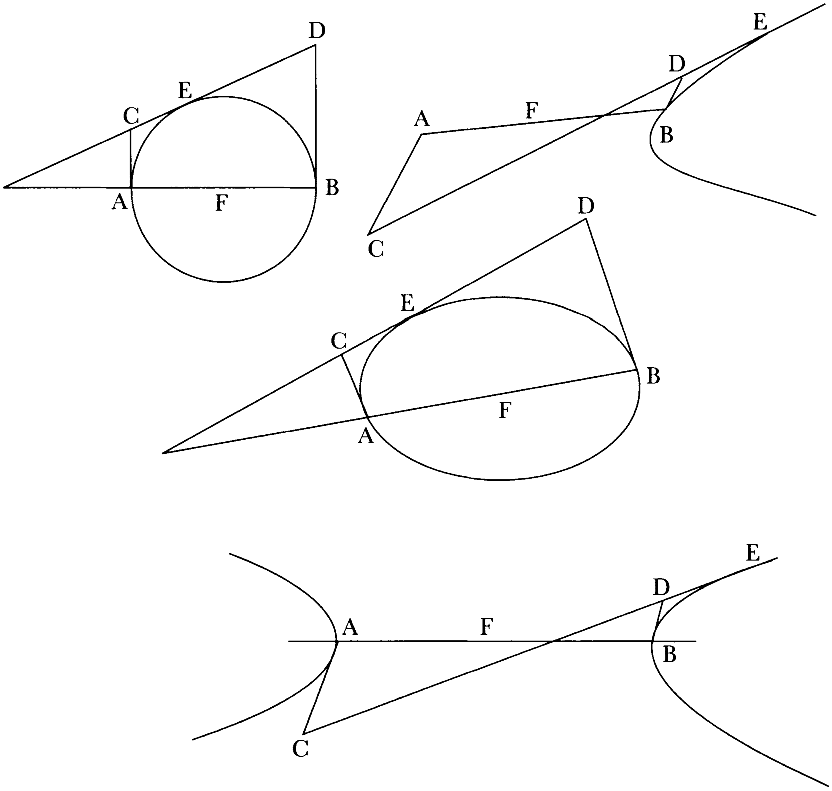

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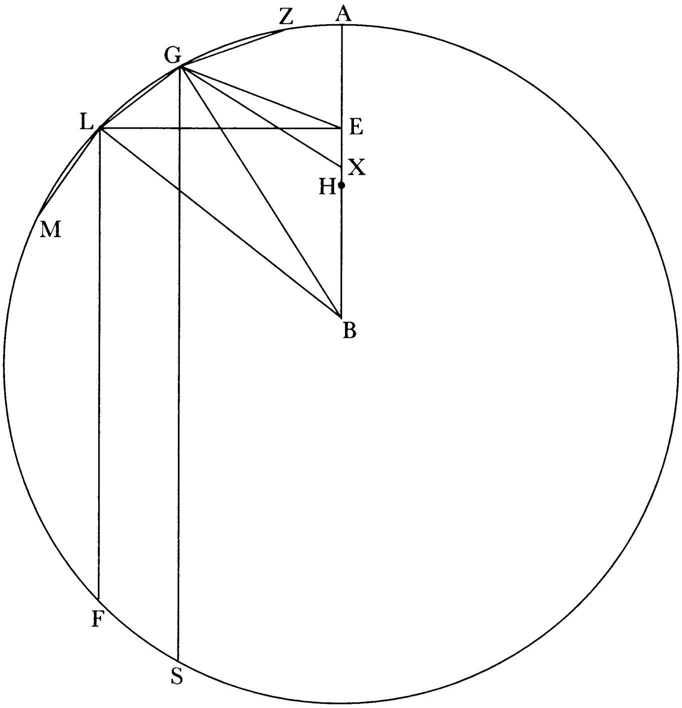

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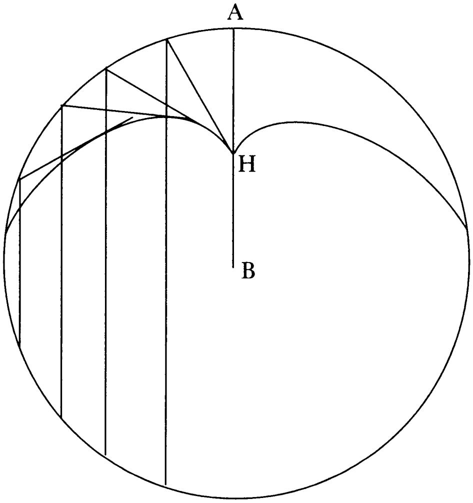

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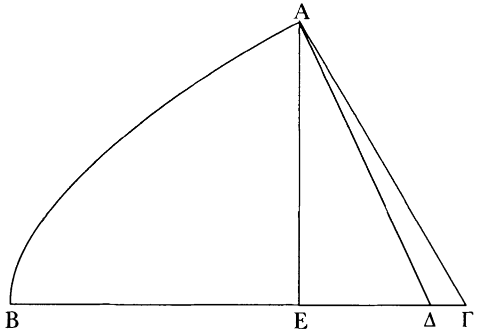

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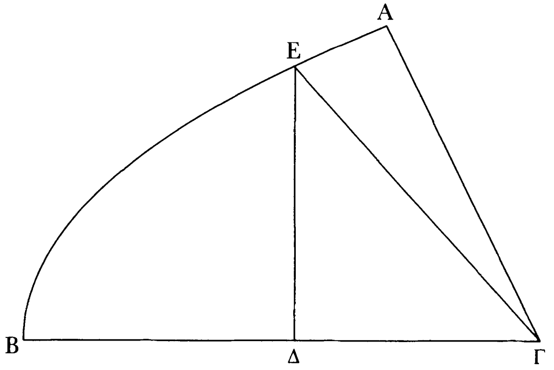

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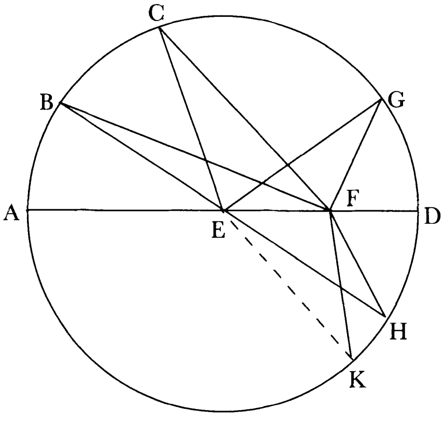

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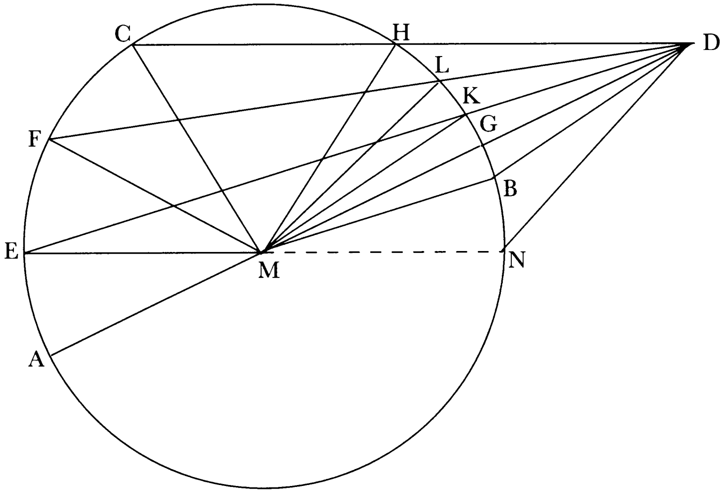

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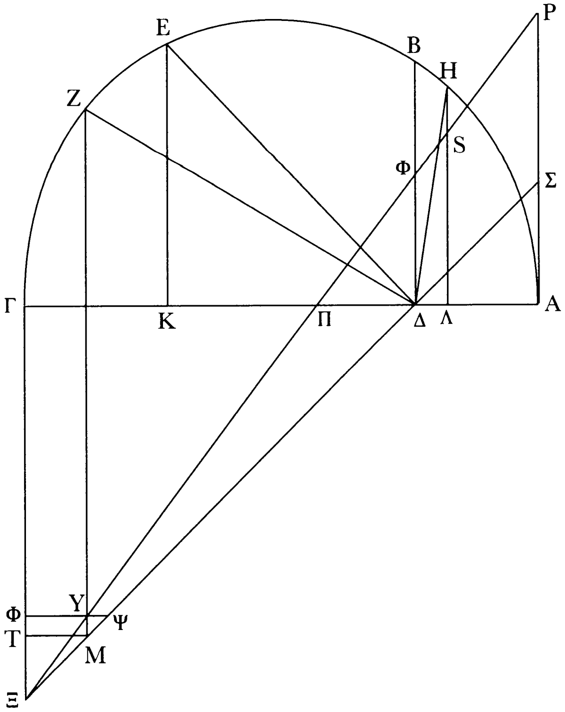

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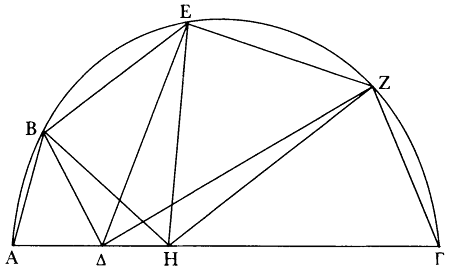

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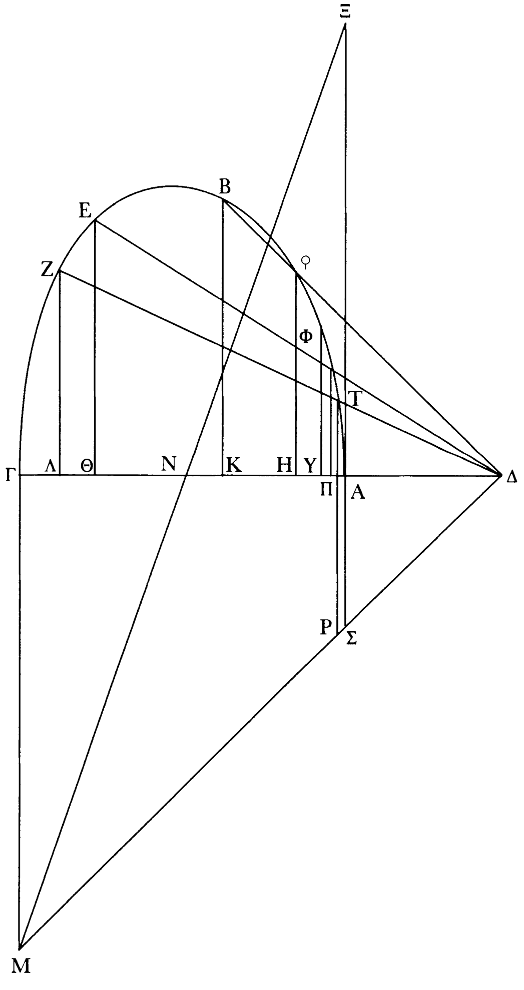

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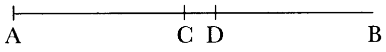

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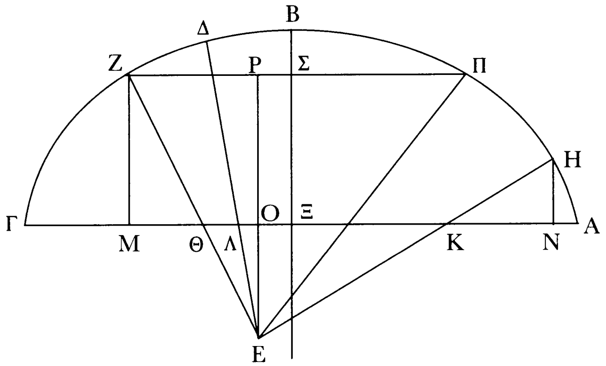

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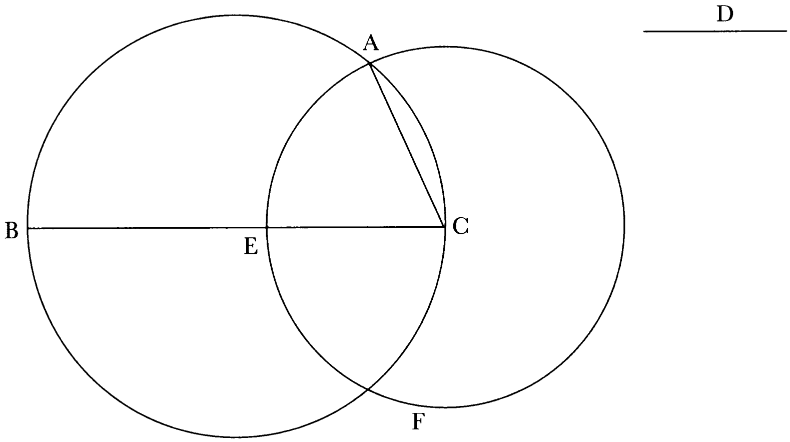

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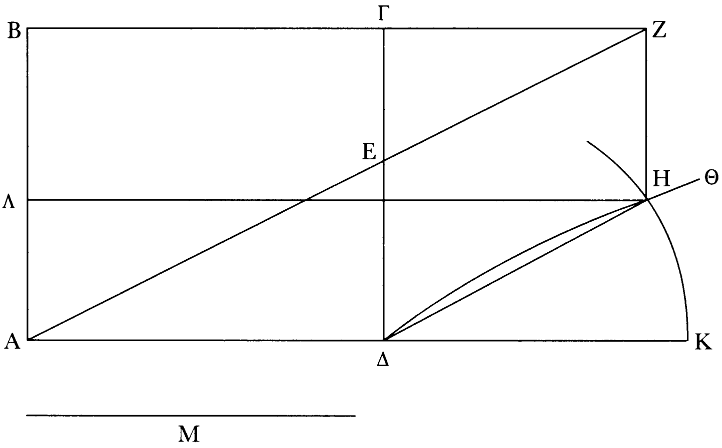

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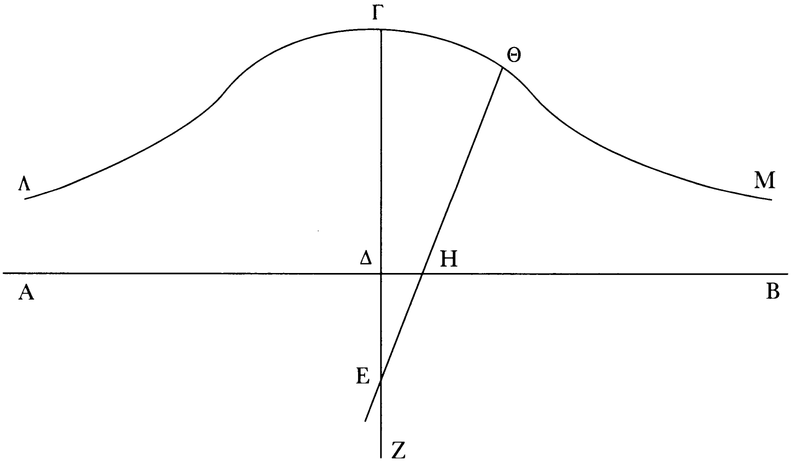

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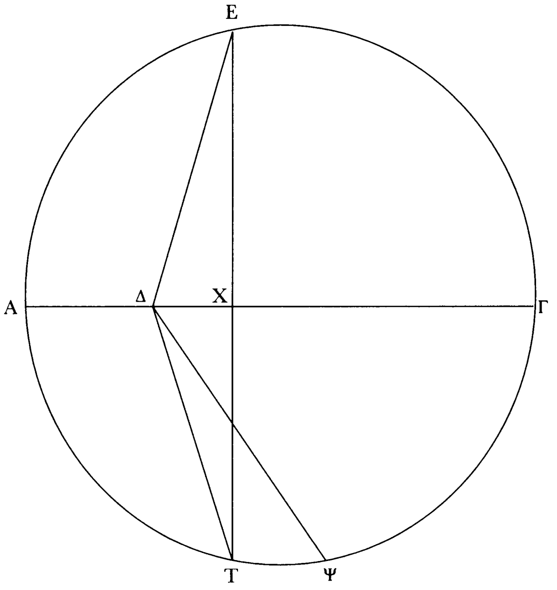

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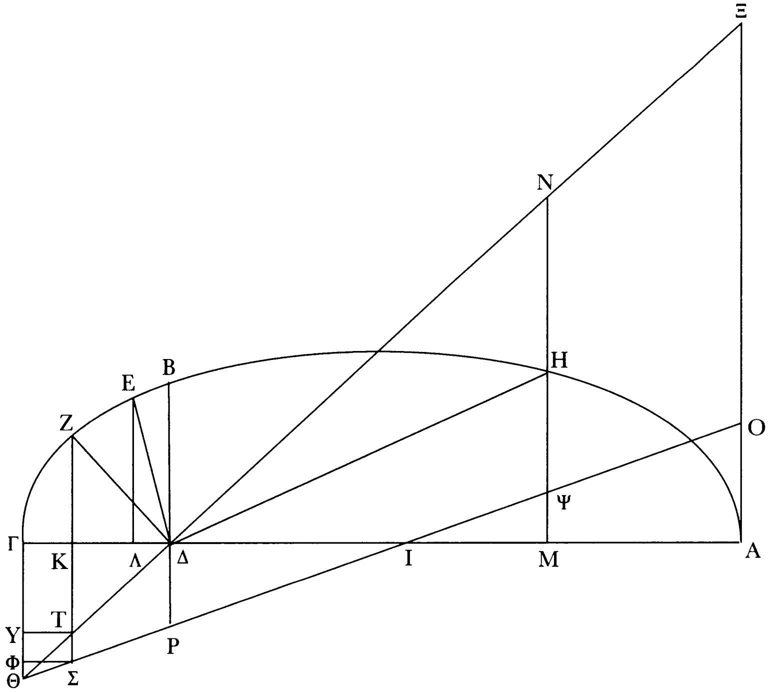
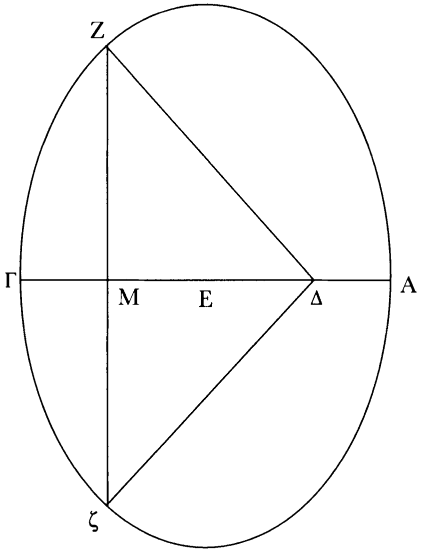


Fig. 29



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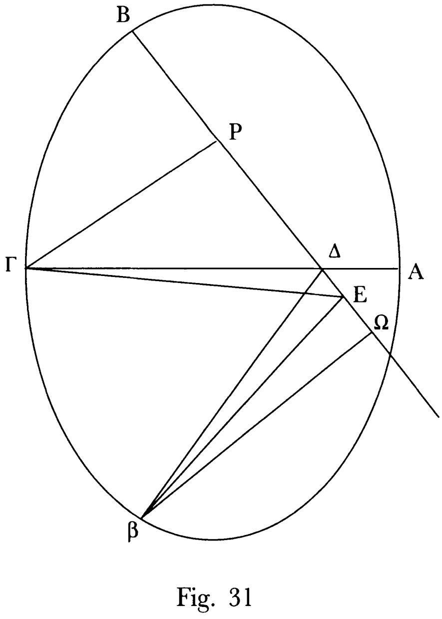

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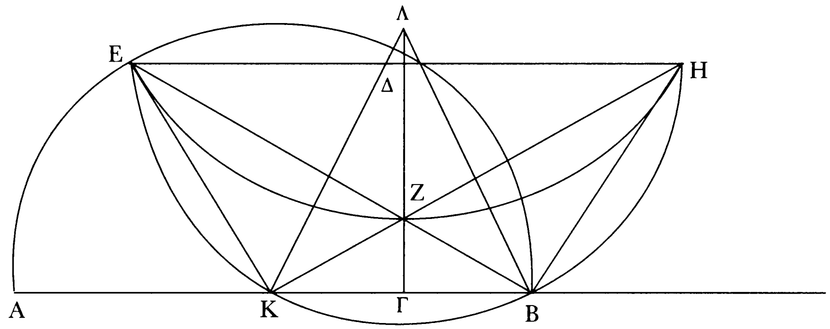

Fig. 32

CHAPTER FIVE

EQUALITY AND SIMILARITY: BOOK VI OF THE *CONICA*

Introduction

The sixth book of the *Conica* merits the shortest chapter in Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*. Indeed, Zeuthen explains at the start of this small nine and a half page chapter that "Für die Aufgabe, die ich mir gestellt habe, nämlich ans Licht zu ziehen, was die Griechen überhaupt von den Kegelschnitten wußten, sowie zu untersuchen, wie sie dieses Wissen erwarben, befestigten und anwandten, *hat Apollonius' 6tes Buch keine große Bedeutung* [emphasis added]."[1] Heath clearly shares Zeuthen's low opinion of Book VI. In his *History of Greek Mathematics*, Book V is given careful and detailed consideration while Book VI is covered in just a little more than one page. And Heath begins that treatment of Book VI, appropriately enough, with the words "Book VI is of much less interest."[2] In this chapter, we shall show that Book VI is undeserving of this kind of disparagement. On the contrary, a careful reading of Book VI reveals much about Apollonius's distinctly geometric approach to the study of conic sections, which in turn yields, among other things, key insights into the meaning and character of Book I, whose importance neither Heath nor Zeuthen nor any other writer on the *Conica* denies.

A glance at Book VI and its contents is enough to see that, in several respects, it stands out among the other books of the *Conica*. For one, it is the shortest of all the extant books, a mere thirty-three propositions. Its organization is exceedingly clear; it is, for example, the only book other than Book I with formal definitions. Its propositions, on the whole, are not very long or difficult. On the face of it, then, Zeuthen is right to say that, for one who has studied Book

[1] H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, p. 384.

[2] Sir Thomas Heath, *A History of Greek Mathematics*, vol. II, p. 167.

I thoroughly, Book VI contains no great mathematical difficulties and that its virtue lies in its being well organized and systematic; it is a book, he says, meant to aid the less sure investigator (der weniger sichere Forscher).[3] But this cannot be the whole story. It is true, Apollonius does not seem particularly discriminating with respect to the whole treatise when he writes to Eudemus at the end of the prefatory letter to Book I that "... with all of [the books] published, *those happening upon them* [emphasis added] can judge them as they see fit (τοῖς περιτυγχάνουσι κρίνειν αὐτά, ὡς ἄν αὐτῶν ἕκαστος αἱρῆται)."[4] Except for this, however, there is no hint that Apollonius ever takes "the less sure investigator" into consideration, and in one instance, at least, Apollonius shows concern that his work be seen only by *able* mathematicians. This is in the letter introducing Book II where Apollonius, writing to the same Eudemus, says, "Go through [the book] then carefully and acquaint those with it *worthy of sharing in such things* [emphasis added]." Even if one assumes that the results in Book VI are truly only easy consequences of propositions in Book I worked out for the "less sure investigator," one needs to ask why Apollonius did not incorporate these results into Book I? Indeed, given the apparently elementary character of Book VI, one wonders why Apollonius did not include its results among the "elementary" books, Books I–IV. The answer, far from being that Apollonius wanted to write a book to prevent the "less sure investigator" from falling into error,[5] is, we believe, 1) that the equality and similarity of conic sections is, for Apollonius, a far more subtle affair than we would like to think, and 2) that the investigation of equality and similarity is necessary for clarifying what is meant by a conic section being "given," so that such an investigation, among other things, aids in the theoretical grounding of analyses using conic sections.[6]

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[3] Zeuthen, *Die Lehre, ibid.*

[4] Unless noted otherwise, all quotations from the first three books of the *Conica* are translated by R. Catesby Taliaferro in *On Conic Sections, Books I–III by Apollonius of Perga.*

[5] Zeuthen, *Die Lehre*, p. 384.

[6] In this connection, Knorr writes, "... Apollonius, like Euclid, intended compilations of this sort to serve the interests of the analytic solution of problems; he makes this claim explicitly in his prefaces and is confirmed by Pappus who assigns to the *entire topos analyomenos* the same role. In the *Conics* the problem-solving aspect is especially prominent in the later books, treating the more advanced portions of the theory" (Knorr, *The Ancient Tradition of Geometrical Problems*, p. 313). Knorr, however, overstates the case. Indeed, Apollonius relates his work in conic sections to

Book VI, therefore, is not an elementary book either in the sense of being directed toward beginners or in the sense of being concerned with the generation, definition, and parts of conic sections; it does not merely elaborate ideas already developed in Book I, but complements those ideas, somewhat in the way Euclid's *Data* complements the *Elements*, as we shall see later.

In our discussion of Book VI, we shall follow, more or less, the structure of the book. As mentioned above, this structure is far more perspicuous here than in the other books of the *Conica*. Book VI begins with a series of definitions that, unlike the definitions in Book I, concern *specifically conic sections and cones*. These definitions include those of *equal* and *unequal*, *similar* and *dissimilar* conic sections; the *base*, *diameter*, and *vertex* of a segment of a conic section or circumference of a circle;[7] *equal* (from their bases up) segments, *unequal* segments, and *similar* segments; conic sections *placed in* a cone and cones *containing* conic sections; *similar* right cones;[8] the *figure* of a section constructed on the axis or on the diameter. The thirty-three propositions which follow these definitions can be divided, roughly, into three groups. The first comprises propositions 1–10 and concerns the equality/inequality of conic sections and segments of conic sections. The second group comprises propositions 11–25 and concerns the similarity/dissimilarity of conic sections and segments. The third group comprises propositions 26–33 and concerns placing conic sections in cones and finding cones containing conic sections.[9] This last

problems in the prefatory letters to Books I, IV, V, and VII, but never does he insist, as Knorr insists, that ". . . the theory of conics has value *principally* [emphasis added] for its use in the investigation of geometrical problems" (p. 328).

[7] Apollonius does not define what is meant by a *segment* of a conic section. Euclid, by contrast, does define what is meant by a segment of a *circle* in *Elem.*III. Definition 6: "A *segment of a circle* is the figure (σχῆμα) contained by a straight line and a circumference of a circle." A possible explanation is that since Apollonius proved in *Conica* I.10 that just as a line joining two points on a circle falls within the circle (*Elem.* III.2) so in a conic section the line joining two points on the section falls within the section, he thought the definition of a segment of a circle was actually applicable to a conic section as well.

[8] This is essentially Euclid's definition 24 in *Elem.*XI.

[9] Propositions 26–27, that parallel cutting planes produce similar sections, form a bridge between the propositions on similarity and the problems which end the book. One might consider them belonging, rather, to the propositions on similarity, that is, to the second group of propositions described above. But in that case, it would be hard to understand the *placement* of these two propositions. For VI.26–27, concerning the similarity of *sections*, follow VI.17–25, propositions concerning similarity of *segments*. Moreover, VI.26–27 do not depend on VI.17–25. Hence, if

group of propositions contains constructions resembling those at the end of Book I but distinguishable from them in certain significant ways. Accordingly, in the first two sections of this chapter we shall discuss, respectively, the equality and similarity of conic sections and shall show that, in these propositions, Apollonius is attacking the problem of the *shape* of conic sections in a very direct geometric way. It should become clear in the course of the discussion that Apollonius is particularly interested, among other things, in showing how these shapes *differ among themselves*, implying that having different συμπτώματα does not make it immediately evident that the corresponding curves have different shapes. In the third section, we shall compare the propositions concerning equality with those concerning similarity. From this we shall see that, while there are obvious analogies among these propositions, equality and similarity of shape are *fundamentally* distinct notions. This, of course, is *not* true in an algebraic setting. The fourth section will focus on the constructions at the end of Book VI and compare them to those at the end of Book I. The difference that will emerge will be connected to a certain willingness, on Apollonius's part, to see the conic sections as independent individual objects. This difference is reflected in Apollonius's referring to conic sections throughout as *given*.

Equality

The first two definitions in Book VI are of *equal* and *unequal* conic sections. Apollonius defines these as follows: "Conic sections which are called *equal* are those which can be fitted, one on another, so that the one does not exceed the other," and, "Those which are said to be *unequal* are those for which that is not so."[10] There is some ambiguity here as to whether Apollonius means the surfaces contained by the curves are to be fitted on one another or the curves themselves. The words, "so that the one does not *exceed* [emphasis added] the other," seems to imply the former interpretation. Clearly,

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Apollonius had intended VI.26–27 to be included in the group of propositions on similarity of sections and segments, one would expect them to appear *before* VI.17, but this is not the case; hence our hypothesis as to their nature.

[10] Unless noted otherwise, all quotations from Book VI of the *Conica* are translated by G. J. Toomer in *Apollonius* Conics *Books V to VII.*

however, if the surfaces coincide so will the curves bounding them and, conversly, if the curves coincide so will the surfaces they contain, assuming, of course, that the surfaces are planes. Apollonius usually avoids the issue by emphasizing both the curves *and* the surfaces. For example, the enunciation of VI.7 is "The perpendiculars drawn from a parabola or hyperbola to its axis, and extended to the other side, cut off from the section on both sides of the axis segments which, when one is applied to another, fit so as not to exceed or fall short of it [*i.e.* they are *equal*],* but do not fit on any other part of the section if placed on it." In the setting out and specification of the proposition, however, Apollonius writes:

> Let there be a parabola or hyperbola ΓΒΑ, with axis ΓΗ. We mark on the section two points, Β, Α, and draw from them two perpendiculars to ΓΗ, and extend them to the other side of the section: these are ΒΖΔ, ΑΗΕ. Let them cut off from the section two segments, ΒΓΔ, ΑΓΕ.
>
> Then I say that curve ΒΓ fits on curve ΓΔ, and curve ΒΑ on curve ΔΕ, *and* [emphasis added]* surface ΑΓΗ on surface ΗΓΕ, and segment ΑΒΓ of the section on segment ΓΔΕ.

Whether it is the curve being fit on the curve or the surface on the surface is, perhaps, moot; what is clear, however, is that *equality* involves fitting something on something else. Presumably, "to fit on," or "to coincide with"[11] was the verb ἐφαρμόζειν in Apollonius's original text. This is the word Apollonius uses in the preface to Book IV when he says, "[Book IV] contains a discussion of the question, in how many points at most it is possible for the sections of cones to meet one another and the circumference of a circle, *on the supposition that they do not coincide throughout* (ἐάνπερ μὴ ὅλαι ἐπὶ ὅλας ἐφαρμόζωσιν) [emphasis added] . . ."[12] It is also the word Euclid uses in his fourth common notion, "*Things which coincide* (τὰ ἐφαρμόζοντα) with one another are equal to one another."[13] Considering the striking resemblance between this common notion and Apollonius's definition of equality among conic sections, a digression on Euclid's notion of equality in the *Elements* is not completely out of place. So, this we shall do.

[11] In the Banū Mūsā text the word is from the root *tabak.*

[12] Heath's translation in *Apollonius of Perga, Treatise on Conic Sections*, p. lxxii.

[13] Heath's translation in *Euclid, The Thirteen Books of the Elements*, p. 155. Unless noted otherwise, all translations from the *Elements* will be Heath's.

The fourth common notion is, of course, not a definition of equality but a basic criterion for it, we would say, *the* basic criterion. Like Apollonius's definition, this criterion involves the thoroughly geometric intuition of equality as *coinciding*; it does not involve the weighing and balancing of some numerical quantity.[14] Euclid uses the common notion most notably in *Elem.* I.4,8 and III.24. Proposition I.4 states, as is well known, that "If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend." In his proof Euclid begins with two triangles, ABC and DEF (see fig. 1) in which the sides AB, AC equal the sides DE, DF, respectively, and the angle BAC equals the angle EDF. The triangle ABC is then *applied* (ἐφαρμόζεσθαι) to DEF,[15] the point A being placed on the point D and the straight line AB

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[14] One suspects that this geometrical notion of equality as coinciding is at work even in the arithmetical books of Euclid's *Elements* where numbers are compared by *measuring* them with a common unit For in measuring a number by a unit, or in general, by another number, one must remove numbers from the measured number each *equal* to the measuring number. Since numbers are always represented in the *Elements* by line segments, one is forced to think of the procedure as removing from the measured number segments *coinciding* with the measuring number.

[15] Much has been made of the method of proof here with its seeming dependence on motion, at least since Jacques Peletier's remarks in his 1557 *In Euclidis Elementa geometrica demonstrationum libri sex* (cited by Heath, *Euclid*, I, 249). In fact, it is not completely obvious that Hellenistic mathematicians found such a proof objectionable. Heath is ambivalent. He writes, ". . . it is clear that Euclid disliked the method and avoided it wherever he could . . . It looks as though he found the method handed down by tradition (we can hardly suppose that, if Thales proved that the diameter of a circle divides it into two equal parts, he would do so by any other method than that of superposition), and followed it, in the few cases where he does so, only because he had not been able to see his way to a satisfactory substitute. But seeing how much of the Elements depends on I.4, directly or indirectly, the method can hardly be regarded as being, in Euclid, of only subordinate importance; on the contrary it is fundamental. Nor, as a matter of fact, do we find in the ancient geometers any expression of doubt as to the legitimacy of the method" (*Euclid*, I, 225). Work by C. M. Taisbak has shown, however, that much of this ambivalence may be removed if one considers the "applying" or "fitting" of one geometrical entity on another in relation to the notion of being "given in position (τῇ θέσει δεδόσθαι)," as set out in Euclid's *Data*, rather than to some type of motion. Taisbak writes, "We are able, by now, to listen to *Euclid's apology* of I.4: This is not *kinêsis*, movement. How can I be moving something which has no place (yet)? The two triangles are not in the Plane when the curtain rises; they are being thought of as having those properties in common. Then the positioning starts, but hesitantly (as can be seen from the imperfect aspect of the verb

on DE. Euclid then says that point B will also coincide with E because AB is equal to DE; the line AC will coincide with DF because the angle BAC is equal to the angle EDF; and, since AC is equal to DF, point C will coincide with point F. He concludes the proof as follows:

> But B also coincided with E; hence the base BC will coincide with the base EF.
>
> [For if, when B coincides with E and C with F, the base BC does not coincide with the base EF, two straight lines will enclose a space: which is impossible.
>
> Therefore the base BC will coincide with EF][16] and will be equal to it.
>
> Thus the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it.
>
> And the remaining angles will also coincide with the remaining angles and will be equal to them, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

*Elem.*III.24, which states that "Similar segments of circles[17] on equal straight lines are equal to one another," is proven along similar lines. Euclid begins with two segments, AEB and CFD (see fig. 2), on equal straight lines, AB and CD, respectively. He then applies AEB on CFD, the point A being placed on C and the straight line AB on CD. As in I.4, Euclid says that point B will then coincide with D since AB is equal to CD, and, with that, concludes that AEB will coincide with CFD. For if not there are three possibilities: either AEB falls within CFD, or AEB falls outside CFD, or AEB falls awry such as CGD in the figure above. The proof of III.23 takes care of the first two possibilities[18] while III.10, that "A circle does not cut

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[ἐφαρμόζεσθαι]): one triangle, DEF, has been put (tacitly) in the plane before ABC starts being put on top of it. Nothing prevents the Geometer from having ABC put wherever he wants to, *e.g.* on top of DEF if it be possible. *This is not movement*, but positioning, instantiation, as seen from the viewpoint of the ancients" ("Movement and Position in the *Elements* and the *Data*," p. 4).

[16] The square brackets are inserted by Heath who sees the passage as an interpolation (see *Euclid*, I, 225).

[17] In *Elem.* III. def.11, Euclid defines *similar segments of circles* as ". . . those which admit equal angles, or in which the angles are equal to one another."

[18] There is a striking analogy between the pair I.7,8 and the pair III.23,24, not the least in the diagrams they employ, the one for straight line figures the other for curves. Regarding this C. Dodgson (Lewis Carroll) writes, "These analogies give to geometry much of its beauty, and I think that they ought not to be lost sight of" (*Euclid and his modern Rivals*, pp. 194–5, quoted by Heath in *Euclid*, I, 261).

a circle at more points than two," takes care of the third. "Therefore," Euclid concludes, "if the straight line AB be applied to CD, the segment AEB will not fail to coincide with CFD also; therefore it will coincide with it and will be equal to it."

Euclid's general strategy in these propositions is transparent: he proves that the two figures involved *coincide* when one is fitted on the other and then, applying common notion 4, concludes that the figures are *equal.* As to the details of the proofs, one should pay careful attention to two things. The first is that when Euclid says in *Elem.*I.4 that point B will also coincide with E because AB is equal to DE; the line AC will coincide with DF because the angle BAC is equal to the angle EDF; and, since AC is equal to DF, point C will coincide with point F, and in *Elem.*III.24 that the point B will then coincide with D since AB is equal to CD, Euclid is not using common notion 4 but its *converse.*[19] He does so without a second thought because the geometric entities involved, lines and angles, are the simplest possible; they are entities for which it is not possible to separate being equal and having the same shape,[20] so that for *these basic* geometric entities common notion 4 and its converse are equally valid and, thus, common notion 4 becomes a definition of equality. The second observation is that it is *from* these lines and angles that the equality of the more complex figures is established. In other words, it is precisely these entities, those for which equality *means* coinciding, that are the "anchors" for the proofs of the propositions just described. But what of things not so simple or basic?

Towards the end of Book I of the *Elements,* the reader learns that beyond line segments, angles, triangles having two sides and an included angle equal, and the like, not all objects that are equal coincide. This is the revelation of *Elem.*I.35, which states that "Parallelograms which are on the same base and in the same parallels are equal to one another." That things can be equal while not coinciding is without a doubt tremendously important in Greek mathematics. It is the theme, for example, dominating the remainder of

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[19] Heath, *Euclid,* I, 250.

[20] Hence, it is in connection to common notion 4, that Proclus defines the phrase "of the same shape" and relates it to such basic entities: ὁμοειδῆ δὲ λέγω οἶον εὐθεῖαν εὐθείᾳ καὶ περιφέρειαν περιφερείᾳ τοῦ αὐτοῦ κύκλου (*In Euclid.* p. 241). In general, objects for which the converse of common notion 4 applies are those that are described as ἴσος τε καὶ ὅμοιος (see Mugler, pp. 228–9). We shall have more to say about this later.

Book I and most of Book II in the guise of the study of application of areas.[21] This, however, does not warrant saying, as Heath does, that "... we are in this proposition [*i.e.*, *Elem.*I.35] introduced for the first time to a *new conception* [emphasis added] of equality between figures. Hitherto we have had equality in the sense of *congruence* only, as applied to straight lines, angles, and even triangles (cf. I.4). Now without any explicit reference to any change in the meaning of the term, figures are inferred to be equal which are *equal* in *area* or in *content* but need not be of the same form."[22] It is plain that in *Elem.*I.35 Euclid develops and expands the idea of equality, of what things can be equal, but he has not introduced a *fundamentally new conception* of equality and he has not slyly replaced equality of figures with the equality of some *new* entity, *content.* There is no need for him to make "any explicit reference to any change in the meaning of the term ..." because he has not changed it. To see this and to get a better grasp of how Euclid *does* understand equality, consider the proof of I.35:

> Let ABCD, EBCF be parallelograms on the same base BC [see fig. 3] and in the same parallels AF, BC; I say that ABCD is equal to the parallelogram EBCF.
>
> For, since ABCD is a parallelogram, AD is equal to BC.
>
> For the same reason also EF is equal to BC, so that AD is also equal to EF [by common notion I]; and DE is common; therefore the whole AE is equal to the whole DF [by common notion 2].
>
> But AB is also equal to DC; therefore the two sides EA, AB are equal to the two sides FD, DC respectively, and the angle FDC is equal to the angle EAB, the exterior to the interior; therefore the base EB is equal to the base FC, and the triangle EAB will be equal to the triangle FDC [by I.4].
>
> Let DGE be subtracted from each; therefore the trapezium ABGD which remains is equal to the trapezium EGCF which remains [by common notion 3].

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[21] This theme includes several figures *together* being equal to another figure (a possibility, we believe, as astonishing for Euclid as that of the equality of non-coinciding figures). Thus, in this reading, *Elem.*I.47 taken together with *Elem.*VI.31 (that "In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle") are important since they show *how* a single figure can be equal to two *similar* figures together. Incidently, Euclid never says in these propositions that the one figure is equal to the *sum* of the other two as modern books do, that is, he does not seem to say that some numerical quantity (*e.g.* area) measuring the one is equal to the arithmetical sum of those measuring the other.

[22] Heath, *Euclid*, I,327.

> Let the triangle GBC be added to each; therefore the whole parallelogram ABCD is equal to the whole parallelogram EBCF [common notion 2].

When Euclid says triangle EAB is equal to triangle FDC, he is clearly relying on I.4 which, as we have seen, depends on common notion 4. These triangles, therefore, are equal in the sense of coinciding. Next, Euclid removes from these equal triangles the triangle DGE. Obviously, DGE coincides with itself and so is equal to itself. Thus, in removing DGE from each of EAB and FDC we are removing equals from equals and common notion 3, "if equals be subtracted (ἀφαιρεθῇ)[23] from equals, the remainders are equal," gives us the authority to say that what remains is equal. From here, we can add to the remaining equal parts the triangle GBC, again equal to itself in the sense of coinciding with itself, and assert that the result is equal on the authority of common notion 2, that "if equals are added (προστεθῇ) to equals, the wholes are equal." From the first four common notions, accordingly, one can reconstruct Euclid's notion of equality as follows: things are equal if they coincide or if they are the result of adding or subtracting coinciding things from coinciding things. Hence, while things need not coincide, they are always ultimately *derivable* from coinciding things. Put another way, even when figures are non-coinciding, "coinciding" ultimately *mediates* their equality. And, in this sense, coinciding still remains the basic criterion for equality and the source of any approach to equality. Apollonius,[24] in taking coinciding as the *definition* of equality of conic sections, keeps his treatment close to this geometrically intuitive source. Indeed, aware of the possibility of being equal *without* coinciding, he emphasizes that "Those [conic sections] which are said to be *unequal* are those for which that [namely, being able to coincide] is not so." So, as with line segments, angles, and circumferences of circles, conic sections can be equal, then, in this *one* way only, by coinciding. This sets conic sections among things whose shape is unambiguous and,

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[23] Mugler uses "to cut off" as the proper English translation of ἀφαιρεθῇ and emphasizes the geometrical sense of the word. Concerning ἀφαιρεθῇ, he writes, "Opération consistant à retrancher d'une grandeur géométrique une grandeur du même ordre, c'est-á-dire une ligne d'une ligne, une aire d'une aire, un volume d'un volume" (p. 89).

[24] The assumption here is that Apollonius's notion of equality in general was not very different from Euclid's. Indeed, we know from Proclus (*In Euclid.*, p. 194) that Apollonius tried to prove the common notions in the *Elements*.

accordingly, makes an investigation of equality of conic sections an investigation of their shape. Of course, the converse is not true: an investigation of shape involves also, and perhaps principally, the notion of similarity, but this we shall discuss later.

Apollonius begins his investigation of equality, first, by showing under what circumstances a parabola is equal to a parabola, a hyperbola to a hyperbola, and an ellipse to an ellipse, and then, by showing that an ellipse is never equal to either a parabola or hyperbola and that a parabola is never equal to a hyperbola. This is the pattern begun in the definitions and continued to a very great extent throughout the book: first, things equal, then, things unequal, similar, then, dissimilar. Let us begin the discussion with things equal. We shall consider, as an example, VI.2 which treats equal hyperbolas and ellipses. Some of what follows has already been said in the second chapter in a slightly different context. The present discussion will stress, especially in light of the account of *Elem.*I.4 and III.24 given above, the thoroughly geometric way in which Apollonius overcomes the difficulty of superimposing non-circular curved shapes.

Proposition VI.2 states that "If the figures[25] that are constructed on the transverse axes of hyperbolas or ellipses are equal and similar, then the sections [themselves] will be equal; and if the sections [either hyperbolas or ellipses] are equal, then the figures constructed on their transverse axes are equal and similar, and their situation is similar."[26] The proof of the first part is by contradiction and runs as follows:

> Let there be two hyperbolas or ellipses, AB, ΓH, with axes AK, ΓΘ [see fig. 4].

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[25] The "figure of the section constructed on the axis or on the diameter" is defined by Apollonius at the start of Book VI (although he *uses* the term frequently, without defining it explicitly, in the first four books, as in II.1: "If a straight line touch an hyperbola at its vertex, and from it on both sides of the diameter a straight line is cut off equal to the straight line equal in square to the fourth of the *figure* [emphasis added], then the straight lines drawn from the center of the section to the ends thus taken on the tangent will not meet the section.") as, "that [figure] [Toomer's brackets] contained by the axis or diameter together with the *latus rectum*."

[26] The square brackets here, as in the proof which follows, are in Toomer's translation. As before, all square brackets in Toomer's translation will be Toomer's. Regarding these square brackets, Toomer says, "I have been liberal with explanatory additions, which are enclosed in square brackets:[so]" (*Apollonius* Conics *Books V to VII*, p. xci). Where the square brackets are ours, we shall follow them, as before, with an asterisk.

Let the figures which are constructed on their transverse axes be equal and similar: these are ΔE, NΛ.

Then I say that the sections AB, ΓH are equal.

Proof of that:

We apply axis AK to axis ΓΘ: then section [AB] will coincide with section [ΓH].

For if that is not so, let a part of section AB not coincide with section ΓH. We mark point B on that part, and draw from it a perpendicular to the axis, BK, and complete the delineation of [the corresponding] rectangle ΔZ. We cut off from ΓΘ a line equal to line AK, namely ΓΘ, and draw from point Θ a perpendicular to ΓΘ, namely ΘH, and complete the delineation of [the corresponding] rectangle NM.

Then lines AE, AK are [respectively] equal to lines ΛΓ, ΓΘ.

Therefore, rect.EK = rect.ΛΘ.[27]

Furthermore, rectangles ΛM, EZ are similar and similarly situated, because they are similar to rectangles ΔE, NΛ [respectively]; and AK = ΓΘ.

Therefore, rect.EZ = rect.ΛM.

And rectangles KE, ΘΛ were [already proved] equal.

Therefore, rect.AZ = rect.ΓM.

And the lines which are equal to them in square are [respectively] BK, ΘH, as is proven in Props. 12 and 13 of Book I.

So when the axis is applied to the axis, line BK will coincide with line ΘH, and point B will coincide with point H.

But it [B] was supposed not to fall on section ΓH: that is absurd. So the whole of section AB will fit on section ΓH.

Now, the first integral part of this proof is the one sentence, "We apply axis AK to axis ΓΘ: then section [AB] will coincide with [ΓH]." Why the axis must be applied to the axis is given by an

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[27] Toomer has "∴ □ EK = □ ΛΘ." In general, in all quotations from Toomer we shall write "Therefore" instead of the symbol "∴," "rect.AB" instead of "□ AB," "sq.AB" instead of "AB[2]", "is similar to" instead of the unusual "|||" (Cajori cites (*A History of Mathematical Notations*, §374) only two writers who use this symbol, E. Budden in his *Elementary Pure Geometry* (London, 1904) and A. H. McDougall in his *The Ontario Highschool Geometry* (Toronto, 1914)). By this, we hope to maintain a certain notational consistency throughout the present work as well as to remain somewhat closer to the original text. As to the latter, Toomer himself admitts (although he does not refer specifically to the symbols mentioned above) that "Medieval Arabic mathematical texts, like their Greek forerunners, are almost completely rhetorical in their exposition. In my translation, for the purposes of clarity, I have adopted a mildly symbolic notation . . ." (*Apollonius* Conics *Books V to VII*, p. xci). Likewise, we preserve Toomer's paragraph divisions but not his many and varied non-paragraph indentations.

obvious analysis derived from a statement in the second half of the proof where the sections are assumed equal and the equality, similarity, and similar position of the figure is to be proved. Apollonius says there that when a hyperbola or ellipse is fit on an equal hyperbola or ellipse, the axis of the one will coincide with the other, "For if it did not coincide with it, then the hyperbola would have two axes, and the ellipse three axes, which is impossible." What we should like to emphasize here, however, is that the answer to the question, how ought one position[28] the one ellipse, say, on the other, is by means of a definite line segment, in this case, the axis. This was also the way Euclid positioned the one circle segment on the other in *Elem.*III.24: ". . . if the segment AEB [see fig. 2] be applied to CFD, and if the point A be placed on C and the straight line AB on CD, the point B will also coincide with the point D, because AB is equal to CD; and, AB coinciding with CD, the segment AEB will also coincide with CFD." Similarly, in *Elem.*I.4, the one triangle ABC was positioned on DEF first so that "A be placed on the point D and the straight line AB on DE . . ." So, as mentioned above, fitting one figure on another is carried out by means of simple geometric "anchors," entities for which equality always means coinciding. In the next part of the proof of VI.2 one sees that proving the equality of two more complex figures is carried out by means of a *chain* of such "anchors." To see how this works, let us look closely at this part of the proof.

Having said, ". . . section [AB] will coincide with [ΓH]," Apollonius begins the proof by contradiction in the usual way, "For if that is not so, let a part of section AB not coincide with section ΓH." What one needs to remember here is that the position of section AB is *still* such that its axis, AΔ, is coinciding with ΓN, the axis of section ΓH, as are the figures of the two sections, since they are built on the axes and are given as equal and similar. Apollonius now, from a point, B, on the noncoinciding part of AB, draws a line, BK, perpendicular to the axis and completes the rectangle, ΔZ. With AK set in place, Apollonius cuts off from the axis of section ΓH, ΓΘ equal to AK, draws ΘH perpendicular to the axis of ΓH, and completes the corresponding rectangle NM. As for the general strategy of the proof, drawing BK and HΘ has the obvious but significant

[28] See Taisbak, "Movement and Position in the *Elements* and the *Data*," p. 4.

role of reducing the problem of looking at noncoinciding points on sections to looking at noncoinciding line segments. The details, however, are revealing. Instead of telling us to complete rectangles EZ and ΛM, which play an explicit role in the proof, Apollonius tells us to complete the rectangles ΔZ and NM, which do not. Moreover, we are never told to *complete* rectangles EZ and ΛM, but only told that they are "similar and similarly situated." Apollonius follows this seemingly obscure course, we think, in order to proceed as much as possible from *already* coinciding elements. Thus, he starts with rectangles ΔZ and NM because KΔ and ΘN are already coinciding line segments; once these rectangles are drawn, points Z and M are given in position (and, of course, coinciding), so that, indeed, EZ and ΛM are seen to be "similar and similarly situated." By the way, Apollonius never tells us to draw the diagonals EΔ and ΛN in the diagrams, from which it is reasonable to conclude that these diagonals serve only to emphasize that the rectangles, EZ, ZΔ, EΔ and ΛM, MN, ΛN are all similar and similarly situated rather than to aid in their construction. In the next stage of the proof, Apollonius shows that rectangles EK and ΛΘ are equal and also rectangles EZ and ΛM are equal. The first two are equal because AE, AK are equal to ΛΓ, ΓΘ respectively and the second two are equal because they are similar and AK is equal to ΓΘ. For both pairs, then, equality means coinciding. From this, Apollonius concludes that rectangles AZ and ΓM are equal. This is true because, in the case of the hyperbola, AZ and ΓM are the result of adding EZ to EK and ΛM to ΛΘ, while in the case of the ellipse, AZ and ΓM are the result of removing EZ from EK and ΛM from ΛΘ; the conclusion, of course, is consistent with the strictest interpretation of Euclid's common notions 2 and 3. In addition, since, as Apollonius emphasizes, EZ and ΛM are not only similar but also *similarly situated* the rectangles AZ and ΓM are equal in the sense of coinciding. Finally, using the συμπτώματα developed in I.12,13, the squares on BK and ΘH must be equal since they are equal to AZ and ΓM, respectively (this is the only place where equality is not by coinciding). And from this, Apollonius concludes that BK and ΘH must coincide, leading to the desired contradiction. Hence, the proof proceeds by a chain of coinciding pairs: from the "figures," rects.EΔ and ΛN, to the axes, AΔ and ΓN (as well as to the *latera recta*, AE and ΓΛ), to AK, KΔ and ΓΘ, ΘN to rects.ZΔ and MN to rects.EZ and ΛM to rects.AZ and ΓM to (by means of the συμπτώματα) sqrs.BK and ΘH to BK and ΘH.

VI.1 follows a similar pattern to prove that two parabolas are equal if and only if their *latera recta* (to their axes) are equal. In this case, also, one parabola is positioned on the other by positioning the axis on the axis (and, accordingly, *latus rectum* on *latus rectum*) and then by means of a chain of coinciding elements one shows that if the parabolas, thus positioned, contain any noncoinciding points one arrives at a contradiction. In both VI.1 and VI.2 the συμπτώματα reveal themselves as crucial links in this chain, though they are not themselves the identifying agents, as we argued in the second chapter.

The complement to these theorems concerning the equality of parabolas, hyperbolas, and ellipses, each among themselves, is VI.3b.[29] The enunciation of this proposition is missing, but Toomer suggests that it was something like "No conic section can be equal to a different kind of conic section."[30] Most likely, Apollonius said explicitly, "No parabola can be equal to a hyperbola, nor can any ellipse be equal to either a parabola or a hyperbola." For the previous propositions seem to say that parabolas are of a kind, as are ellipses and hyperbolas, while it is the point of this proposition to say these three are, indeed, all *different* unmixable kinds. The placement of this proposition supports, we think, this interpretation. In this connection, one notes that there is no logical dependence of VI.3b on any of the previous propositions in Book VI; in this sense, it could well have started the book. But there is a dependence in another sense. Look at the beginning of the proof, which considers only the case of a parabola and a hyperbola (Apollonius dismisses the cases of an ellipse and a parabola or a hyperbola as obvious since the ellipse is bounded and the other curves are not):

> Let there be a parabola [see fig. 5], ΑΒΓ, and a hyperbola ΗΙΚΝ.
> Then, if possible, let it [ΑΒΓ] be equal to it [ΗΙΚΝ], and let the axes of the sections be ΒΖ, ΚΜ, and let the transverse axis of the hyperbola be ΚΘ, and let lines ΒΕ, ΒΖ be equal to lines ΚΛ, ΚΜ [respectively]. We draw from the axes perpendiculars ΑΕ, ΔΖ, ΙΛ, ΗΜ.
> Now the section fits on the section, because it is equal to it, and [hence] points Ε, Ζ, Α, Δ coincide with points Λ, Μ, Ι, Η [respectively].

[29] In Toomer's translation.
[30] Toomer, *Apollonius* Conics *Books V to VII*, p. 577.

How does Apollonius know that if ABΓ and HIKN are equal then they will coincide only in such a way that their axes will coincide and, hence, points E, Z, A, Δ will coincide respectively with points Λ, M, I, H? A possible answer is the argument in the second part of VI.2 (and in the second part of VI.1 concerning the parabola), cited above, that, otherwise, a hyperbola would have two axes. One cannot rule this out completely, but one would expect, if it were such an argument, that Apollonius would restate it explicitly or allude to it, as he does in VI.7,8. As it is, he simply assumes the sections coincide this way as if it were the most obvious fact in the world. This leads us to believe that Apollonius was not looking at an isolated argument buried within the previous propositions, but at all these propositions taken together as a whole; the previous propositions and their proofs say that if one is speaking about equal parabolas or about hyperbolas or ellipses, this is how they are fitted on one another. In this reading, the beginning of VI.3b can be interpreted as saying "Suppose parabola ABΓ be equal to hyperbola HIKH. Then they are of the same kind, either both parabolas or both hyperbolas, and we can fit one on the other, therefore, as we did in VI.1, in the one case, and as in VI.2, in the other, *i.e.* axis on axis." The proof then shows that, in fact, they cannot be so fitted and, *therefore*, the parabola is not equal to the hyperbola. The focus on how the sections are to be fitted on one another emphasizes that VI.3b does not separate hyperbolas and parabolas by an ancillary property but through their very incapacity of coinciding (a result sharpened significantly in the last proposition of this group, VI.10), that is, Apollonius separates the sections *in their most basic geometric aspect*, according to their shape. As we shall see in the next section, when Apollonius takes up the subject of similarity and conic sections, he follows precisely the same pattern of establishing kinds (by providing a criterion for comparing individuals of a kind) and then showing that the kinds are different.

The first few theorems of Book VI treat the conic sections in a somewhat general way, through which we know how to compare parabolas, for example, and that parabolas and hyperbolas are different shapes, in the sense that one can never coincide with the other. The remaining theorems in this part of Book VI, the part on equality and inequality, concentrate on equal and unequal *segments* of conic sections. In this investigation, one sees how equality is used as a

probe for studying individual conic sections in order to give a more refined picture of their shape.

The first theorems in this group, VI.4,5 say that "If there is an ellipse, and a line passes through its center such that its extremities end at the section, then it cuts the boundary of the section into two equal parts. And the surface is also bisected [by it]." VI.4 proves the special case when the diameter is an axis and VI.5 proves the general case. Apollonius, clearly, sees the symmetry property expressed in this proposition as a matter of matching or coinciding parts of an ellipse (rather than the kind of transformational property that symmetry has become in modern times).[31] Indeed, in the proof of VI.5 one sees the same "cutting and rearranging" of coinciding figures that characterizes the proof of *Elem.*I.35:

> Furthermore, we do not make AB [see fig. 6] one of the axes, and let the axes be ΓΔ, KΛ, and we draw two perpendiculars [to the axis], AE, BH: then the curve ΓΑΔ fits on curve ΓΖΔ, as was proven in the preceding proposition, and point Z coincides with point A, and surface ΑΓΕ coincides with surface ΓΖΕ.
>
> Furthermore, [curve] ΚΓΛ coincides with [curve] ΚΔΛ, and ΕΘ coincides with ΘΗ, and line EZ with line BH, because line ΕΘ is equal to line ΘΗ, and line EZ to line BH.
>
> And surface ΓΕΖ coincides with surface ΔΗΒ.
>
> So surface ΑΓΕ coincides with ΒΔΗ.
>
> Therefore it [ΑΓΕ] is equal to it [ΒΔΗ], and [hence] curve ΑΓ is equal to curve ΔΒ.
>
> Furthermore, [Δ] ΑΕΘ equals [Δ] ΘΒΗ.
>
> So [surface] ΑΓΘ equals [surface] ΘΒΔ, hence the remainder, [curve] AK, equals the remainder [curve] ΒΛ.
>
> And [hence] curve ΑΚΔ equals curve ΓΛΒ. So the whole of surface ΑΚΔΒ equals the whole of surface ΑΓΛΒ, and curve ΑΚΔΒ [equals] curve ΑΓΛΒ.

Propositions VI.4,5 make the analogy between the diameter of a conic and that of a circle, hinted at in Book I, crystal clear. Recall Euclid's definition of the diameter of a circle (*Elem.*I, def.17): "A *diameter* of the circle is any staight line drawn through the centre and terminated in both directions by the circumference of the circle, and

[31] We should point out that, interestingly enough, there is no term in Greek mathematics for what we ordinarily mean by symmetry. Indeed, the word σύμμετρος refers to geometric magnitudes having a common measure, that is, "commensurable," rather than to "symmetry."

such a straight line also bisects the circle." The close similarity between the wording of this definition and that of VI.4,5 is striking and seems to point, as we have said before, to a conscious effort on Apollonius's part to compare the properties of conic sections and circles. But be that as it may, VI.4,5 begin a series of propositions[32] concerning the symmetry-like property in which parts of a conic section fit on other parts of the same conic section, in the same way that the part of a circle on one side of a diameter fits on the part on the other side. Except for VI.4,5, the propositions in the series discuss, in particular, parts on either side of the *axes* of conic sections. One is tempted, therefore, to think of these propositions as unnecessary since the very definition of an axis implies that for any point on a conic section there is another point such that the chord connecting the points is perpendicular to the axis and bisected by it. But one must not give in to this temptation. Apollonius is concerned with the shape of conic sections in terms of their parts, and a continuous part of a conic section is not a collection of points; he wants to show whole parts coinciding with whole parts rather than all points on one part coinciding with all points of the other (of course, in proofs such as VI,1,2,4 Apollonius arrives at a contradiction by assuming a part of a conic section *does not* fit on another and then showing that an arbitrary point on the one must necessarily fit on the corresponding point on the other).

The key lemma in this series of propositions comparing parts of a conic section is VI.6: "If there is a conic section, and part of it coincides with another part of another section so as to fit on it, then the [first] section is equal to the [second] section." Although this lemma refers to two conic sections, it can equally refer to copies of the *same* section so that it can be used to compare parts of a *single* conic section. Apollonius is not explicit about this, but in propositions VI.7,8[33] he applies a segment of a conic section to another seg-

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[32] It is curious that in this series of propositions there is no counterpart to VI.4,5 for the hyperbola. A possible explanation for the omission is that the "cutting and rearranging" procedure used in VI.4,5 could not be used here since the pieces of the hyperbola "cut" and "rearranged", unlike those of the ellipse, would be unbounded. Indeed, for the unbounded surface contained by the hyperbola, Apollonius could not say without qualification that ". . . the surface is also bisected by [the diameter]."

[33] Toomer does not cite this proposition as being used in VI.8 (see Toomer, *Apollonius* Conics *Books V to VII*, p. lxiv). This is clearly wrong. For at the end of VI.8, Apollonius writes, "Then it will necessarily follow, as it did in the preceding propositions, that the ellipse would have more than two axes: that is absurd."

ment, assumed equal to the first, and then concludes that the parts contiguous to these segments must also coincide. We cannot see any other way of understanding this than that Apollonius effectively positions mentally a copy of the conic section over the original so that the segments, assumed equal, coincide. Consider proposition VI.7, for example. It states that "The perpendiculars drawn from a parabola or hyperbola to its axis, and extended to the other side, cut off from the section on both sides of the axis segments which, when one is applied to another, fit so as not to exceed or fall short of it, but do not fit on any other part of the section if placed on it."[34] The first part of the proof is modeled after the proof of VI.4, while the second part, that the segments cut off by the perpendiculars to the axis are not equal to any other segment is the part relying on VI.6. Thus, having shown that the segments AB and ΔE cut off by perpendiculars BΔ and AE are equal, Apollonius sets out to show that no other segment in the parabola or hyperbola will equal these. He writes:

> So let segment ΘK [see fig. 7] be another segment which is not cut off by these two perpendiculars.
>
> Then I say that segment ΔE, if applied to it [ΘK], will not fit on it.
>
> For if that is not so, and if possible, let it fit.
>
> Then, when ΔE is applied to KΘ so as to fit on it, curve ΓΔ will coincide with the segment which is contiguous to segment ΘK, as is proven in the preceding proposition.
>
> And point Γ of segment ΓΔE will fall on a place different from its position on segment KΘΓ, because KΘΓ is not equal to segment ΓΔE; and axis ΓH will fall on a place different from the position it has [now].
>
> So the parabola or hyperbola gets two axes: that is absurd.
>
> So segment ΔE does not coincide with segment ΘK. [Toomer has mistakenly ΔK].

Toomer takes the "preceding propositions" to be only the argument cited above from VI.2 (Toomer, note 32, p. 578). But Apollonius must also have had in mind the argument at the end of VI.7 (quoted below) which uses VI.6 to set up a situation where the argument from VI.2 may be applied.

[34] VI.8 concerns the analogous property in the case of the ellipse: "[In] every ellipse, perpendiculars which are drawn to the axis and extended in a straight line to the other side of it cut off from the section on either side of the axis segments which fit when one is applied to another; and if they are applied to the segments cut off by the perpendiculars whose distance from the center towards the other side is equal to the distance of the perpendiculars drawn [above], they will fit on them, but will not fit on [any] other segment of the section."

So VI.6 and (its use in) VI.7,8 imply how comparing whole sections, parts of distinct sections, and parts of the same section are interrelated. The last two propositions, VI.9,10, in the section of Book VI concerning equality and inequality, highlight this three part relationship. VI.9 states: "[In] equal sections, those parts of them at equal distances from their vertices[35] will fit one on another, and those [parts] not at equal distances from their vertices will not fit one on another." VI.10 states: "[In] sections which are unequal, no part of one of them will fit on a part of another." It is reasonable to think of VI.9 as the last proposition in the series beginning with VI.4. Yet VI.9 is an almost trivial corollary to VI.7,8 and VI.10 is only a restatement of VI.6 in contrapositve form.[36] Moreover, neither VI.9 nor VI.10 is used in any of the remaining theorems of the book. This leads one to think that, instead of VI.9 being the last in the series starting with VI.4 and of VI.10 being an isolated proposition tagged on to the previous nine, these two propositions must be taken together as a partial *summary* of the previous propositions. Like VI.6, both VI.9 and VI.10 refer to separate conic sections, and they do so in terms of *parts* of individual conic sections. One might say that VI.9 speaks about *corresponding* parts of equal conic sections. Thus, in equal conic sections parts at equal distance from the vertices *correspond.* And, in a single conic section, parts at equal distance from the vertices also are corresponding parts in the sense that a copy of the conic section may be mentally placed on the original such that these parts will coincide. We should add, there is evidence of this type of thinking elsewhere in Greek mathematics, albeit from a much later period. According to Proclus, Pappus proved that the angles at the base of an isosceles triangle are equal in the following way:

> Let ABC [see fig. 8] be isosceles with side AB equal to side AC. Let us think of this triangle as two triangles and reason thus: Since AB is equal to AC and AC is equal to AB, the two sides AB and AC are equal to the two sides AC and AB, and the angle BAC is equal to the angle CAB (for they are the same); therefore all the corresponding parts are equal, BC to CB, the triangle ABC to the triangle ACB, the angle ABC to the angle ACB, and angle ACB to ABC. For these

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[35] Clearly, Apollonius means the vertices related to the axes.

[36] In fact, VI.10 is also almost identical to IV.24 in wording, while the proof of IV.24 is virtually identical to that of VI.6.

are angles subtended by the equal sides AB and AC. Hence the angles at the base of an isosceles [*sic*] are equal.[37]

One might say that VI.1,2 are also about corresponding parts: in equal parabolas the *corresponding latera recta* (with reference to the axis) are equal; in equal hyperbolas or ellipses the *corresponding* figures (with reference to the axis) are equal and similar. But, of course, we have followed Apollonius, a little surreptitiously perhaps, in moving from "segments" to "parts."[38] The parts of a conic section *are*, for Apollonius, its segments; the parts belong to the conic section in a very direct way, just as the base angles of an isosceles triangle belong to the triangle in a direct and immediate way. So, while in VI.1,2 we have vitally important criteria for equal conic sections in terms of simple geometric entities associated with them, criteria which, as we have said, set the conic sections into kinds, by the end of these propositions on equality we are looking at the sections themselves and seeing, when they are equal, *precisely what* in them is equal. Similarly, in VI.10, we see *precisely what* is unequal in unequal sections, having found out in VI.3b only that the kinds of conic sections are, in fact, unequal. In this sense, this part of Book VI, with its interplay between the comparison of whole sections, parts of distinct sections, and parts of the same section, ends with a direct assault on the shape of conic sections, that is the conic sections as they appear freshly cut, as it were, from the cone. Incidentally, this is not the only place where Apollonius speaks of the shape of a geometric object in terms of its parts. Proclus, in discussing Euclid's fourth definition in the *Elements*, that "A straight line is a line which lies evenly with the points on itself," mentions that Apollonius proved in his work *On the Cochlias* that a cylindrical helix is *homoeomeric* (ὁμοιομερής, literally, "having like parts"), that, just as with a straight line, any part of the helix is such that it will coincide with any other part (πάντα τὰ μέρη πᾶσιν ἐφαρμόζει).[39] In asking what parts of conic

[37] Proclus, *In Euclid.*, pp. 249–250 (translation taken from Glenn R. Morrow, *Proclus: A Commentary of the First Book of Euclid's* Elements, pp. 194–195).

[38] In the Arabic text, "segments" is *kataa*, things cut-off, while "parts" is *janazaa*, portions.

[39] Proclus, *In Euclid.*, p. 105. Knorr, in his *Ancient Tradition of Geometric Problems* says regarding the *cochlias* that ". . . it is clear that Apollonius' study of this curve was an extension of prior work by Nicomedes and Archimedes. For the latter, the primary interest lay in its use for solving the circle quadrature in connection with

sections coincide or do not coincide, then, one might think of Apollonius as asking to what extent are conic sections like shapes such as lines, circles, and helices.

In the discussion above, we have emphasized: 1) that Apollonius defines equality in the strictest geometric manner that he could, namely, equality as coinciding. In this, his approach is consistent with Euclid's notion of equality, which, contrary to Heath's view, was thoroughly based in the geometrically intuitive procedure of superimposing[40] one figure on another. 2) Having defined the equality of conic sections in this way, Apollonius puts conic sections in the camp of such things as rectangles, parallelograms, and isosceles triangles, things whose *shape* can be spoken of unambiguously and, thus, things whose *shape* can be explored. With this, the conic sections are separated into kinds and the kinds are shown to be unmixable. Moreover, this separation is based only on the inability of their coinciding, rather than on some attendant property of the conic sections, even the principal one, the σύμπτωμα. 3) The consideration of shape is in terms of coinciding parts of conic sections. This is the approach used in an least one other work on curves by Apollonius, *On the Cochlias.* We also mentioned above that the study of shape is only partly in the domain of equality; one needs also to treat similarity. To this we shall now turn.

Similarity

Unlike equality, one cannot find a general notion of similarity from which Apollonius's definition springs. Zeuthen thinks that there *was* some generally accepted notion underlying the various special cases, although it was not made explicit. He says, "In Wirklichkeit nämlich haben alle diese Fälle etwas gemeinsam, *was man nicht aussprach* [emphasis added], für dessen Erkennung man aber einen sicheren praktischen Blick deutlich zeigte, indem man jedesmal gerade den Figuren, die dieses gemeinsam hatten, und keinen anderen den Namen

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the spiral and the quadratrix. For Apollonius, by contrast, topological, rather than metrical, properties of the curve were the principal concern" (p. 297). By saying Apollonius's interest in the *cochlias* was "topological," we assume Knorr means, more simply and less anachronistically, that Apollonius was interested in the *shape* of the *cochlias*.

[40] See note 15.

'ähnlich' beilegte."[41] We find this view, as tempting as it is, hard to accept. For it is the mark of Greek mathematics generally to spell out and make precise even the most "obvious" notions. Certainly, this is true of Euclid who does not hesitate to tell us such things as "The whole is greater than the part,"[42] and who does not think it should be left unsaid how we are to think about *equality*, as we saw above. Unable to find a general notion of similarity in extant Greek works, therefore, one ought to put greater weight on the possibility that there was no such notion than on the possibility that there was one but it was left unsaid.

In fact, what one finds in the *Elements*, at least, is *different* definitions of similarity for *different* kinds of figures. Thus, in *Elem.*III, def.11, we have *similar segments of circles* being ". . . those which admit equal angles, or in which the angles are equal to one another;" in *Elem.*VI, def.1, we have *similar rectilineal figures* being those that ". . . have their angles severally equal and the sides about the equal angles proportional;" in *Elem.*XI, defs.9,24, we have *similar solid figures* being ". . . those contained by similar planes equal in multitude" (def.9), and *similar cones and cylinders* being ". . . those in which the axes and the diameters of the bases are proportional" (def.24).[43] It is clear that when Euclid uses the word *similar* (ὅμοιος) in all these definitions, he is not using it accidentally; each definition is simply a different answer to the one question, what makes figures have the same *shape?*. Accordingly, when Euclid defines in the *Data* what it means for rectilineal figures "to be given in form (τῷ εἴδει δεδόσθαι)" or, as one might say, "given in *shape*,"[44] the definition is very close to his definition of similar rectilineal figures: "Rectilineal figures are said to be *given in form* in which the angles are given and the ratios of the sides are given" (*Data*, def.3). What is lacking, then, is a single criterion, a single guiding principle, for the answer of this question of shape. One *can* find, on the other hand, some common elements in these definitions. In the definition of similar cones and cylinders (*right* cones and *right* cylinders by Euclid's definition of them), one considers only the ratio of the axis to the diameter of the base: two cones or cylinders will

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[41] Zeuthen, *Die Lehre*, p. 386.

[42] *Elem.*, Common Notion 5.

[43] The definition of similar right cones is, incidentally, identical to Apollonius's definition of the same.

[44] See Mugler, p. 161.

be similar if and only if this ratio is the same for both cones or cylinders and that will be true if and only if the right triangles generating the cones or the rectangles generating the cylinder[45] are *similar* (in the sense of *Elem.*VI, def.1). The introduction of similar rectilineal figures to describe the similarity of some other type of figure is also prominent *and explicit* in the definition of similar solid figures. It is especially striking here since the definition is far from obvious; indeed, Simson showed that the definition for *equal and similar solid figures* (XI. def.10), which differs from XI. def.9 only in adding that the similar plane figures are not only equal in multitude but also equal in magnitude, is not even true in general.[46] But, alas, one does not find the same reduction to rectilineal figures in the definition for similar segments of circles. Although Euclid does define this similarity in terms of an angle, which might link it to rectilineal figures, the angle he chooses is that *in* the segment[47] and this angle is not immediatly associated with a single triangle (or other figure) that can be compared in different segments. It is curious, in this regard, that Euclid did not choose to define similar segments in terms of the angle at the center of the circle subtended by the base of the segment, for then one would have a similar triangle in similar segments, namely, that contained by the sides of this central angle and the base of the segment. A possible reason for Euclid's choosing the angle in the segment rather than the central angle just described is precisely that it is *in* the segment; if the segment is taken as an independent geometrical object (indeed, it is described by Euclid as a *figure* (σχῆμα)), the angle in the segment is still immediately given, whereas the central angle demands the construction of another figure, namely, the circle containing the segment. That the definition of similarity of figures is in terms of elements *in* the figure, that is, elements *immediately* associated with the figure is something one *can* say is common to the various definitions of similarity in the *Elements*. So,

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[45] Euclid defines the cone in the following way: "When, one side of those about the right angle in a right-angled triangle remaining fixed, the triangle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *cone*" (*Elem.*XI, def.18), and the cylinder thus: "When, one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a *cylinder*" (*Elem.*XI, def.21).

[46] See Heath, *Euclid*, III, pp. 265–267.

[47] The *angle in a segment* is defined by Euclid in *Elem.*III, def.8.

in the *Elements* we find different definitions of similarity tailored for different situations; they share some common themes but no clear common origin.

The similarity of conic sections demands yet another distinct definition. Apollonius's definition is as follows: ". . . *similar* [conic sections] are such that, when ordinates are drawn in them to fall on the axes, the ratios of the ordinates to the lengths they cut off from the axes from the vertex of the section are equal to one another, while the ratios to each other of the portions which the ordinates cut off from the axes are equal ratios." Thus, for example (see fig. 9), the ellipses ABC and abc having axes AF and af, respectively, are similar in which ordinates such as BG, CH, bg, ch are such that 1) BG:DA:: bg:da, CH:EA::ch:ea (or, equivalently, BD:DA::bd:da, CE:EA::ce:ea) while, at the same time, 2) AD:ad::AE:ae. Whether Apollonius invented this definition for the similarity of conic sections or reworked or simply appropriated an already existing definition is not easy to determine. One could argue, in fact, that Apollonius took an already existing definition, since his definition is in terms of the axis only, pointing to a pre-Apollonian knowledge of conic sections in which the only diameter known was the axis. Whatever the case, it is clear that there *were* definitions before Apollonius. Archimedes, for example, mentions that all parabolas are similar (as Apollonius proves in VI.11) in the preface to *Conoids and Spheroids* and gives a condition for similar ellipses, as if it were well known, at the end of proposition 14 of that same work. This condition is that two ellipses will be similar when the ratios sq.DB:rect.AD,DC and sq.HF:rect.EF,FG are equal (see fig. 10) where AC and EG are the axes of the ellipses and DB and HF are drawn perpendicular to AC and EG, respectively. Toomer refers to this condition *itself* as Archimedes's definition of similarity.[48] Archimedes, however, does not state it as a definition, and it may well have been only a useful criterion for similarity proven from a different definition of similarity. Indeed, Apollonius proves in VI.12 that two ellipses will be similar when the figures constructed on the axes are similar. These figures are similar when the ratios between the upright sides and axes are equal. But by *Conica*, I.21, this ratio is equal to the ratio sq.DB:rect.AD,DC in any ellipse and for any perpendicular DB (a fact which plays an important part in

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[48] Toomer, *Apollonius* Conics *Books V to VII*, p. lxii. See also Zeuthen, *Die Lehre*, pp. 386–387.

Apollonius's proof of VI.12). Hence, Archimedes's "definition" can be thought of as an early version of *Conica*, VI.12.

One might ask, nevertheless, why Apollonius did not adopt something akin to VI.12 as his definition of similarity. After all, it is not *impossible* to conceive of a form of VI.12 as a definition of similar ellipses and hyperbolas, as Toomer does, in effect, by considering Archimedes's statement at the end of *Conoids and Spheroids*, 14 a definition. Indeed, a definition as this would have the advantage of reducing the similarity of conic sections (at least two of them) to the similarity of *rectilineal* figures, which, as we saw above, Euclid was inclined to do. Apollonius evidently appreciates the importance of this, though he shows it by proving VI.12 *as a proposition* instead of presenting it as a definition. The obvious objection to such a definition is that it could not be used for the parabola since the parabola has no "figure." It is perfectly reasonable, then, that Apollonius should choose a definition good for all conic sections rather than have one definition for similar parabolas and another for similar hyperbolas and ellipses. But even while rejecting a definition based on VI.12 or, equivalently, Archimedes's "definition," one could imagine Apollonius's *incorporating* elements of it in his own definition. In particular, if one takes Archimedes's criterion as a definition of similarity, the proportions BD:DA::bd:da, CE:EA::ce:ca *imply* the proportion AD:ad::AE:ae in Apollonius's definition. Consider, for example, two ellipses, Abc and ABC (see fig. 11). Suppose Abc and ABC are similar (by Archimedes's "definition"), and let them be positioned in the plane so that one of their vertices and their axes, Af and AF, coincide. Let bd and ce be arbitrary ordinates to the axis Af of ellipse Abc. Draw lines Ab and Ac and extend them to B and C, respectively, on ellipse ABC. Let BD and CE be drawn perpendicularly to the axis of ABC, AF, that is, let BD and CE be drawn ordinatewise to AF. Then, we have immediately (by *Elem.*VI.4),

bd:dA::BD:DA and ce:eA::CE:EA.

Moreover, B and C are clearly the only points (on this side of the axis) from which ordinates may be drawn satisfying the proportion (for otherwise, a line would cut the ellipse in more than two points contrary to *Conica*, I.10), which means that the construction given provides an *arbitrary pair* of ordinates satisfying the proportion. Now, suppose we have, in accordance with the Archimedean condition,

sq.bd:rect.Ad,df::sq.BD:rect.AD,DF::sq.ce:rect.Ae,ef::sq.CE:rect.AE,EF.

Then also,

sq.bd:sq.BD::rect.Ad,df:rect.AD,DF and
sq.ce:sq.CE::rect.Ae,ef:rect.AE,EF.

But, sq.bd:sq.BD::sq.Ad:sq.AD and sq.ce:sq.CE::sq.Ae:sq.AE.
So that,

sq.Ad:sq.AD::rect.Ad,df:rect.AD,DF
sq.Ae:sq.AE::rect.Ae,ef:rect.AE,EF.

From this it follows that,

Ad:AD::df:DF and Ae:AE::ef:EF.

This, of course, implies Ad:df::AD:DF and Ae:ef::AE:EF, or Ad:Af:: AD:AF and Ae:Af::AE:AF, so that, also, Ad:AD::Af:AF and Ae:AE:: Af:AF.

And, therefore, finally,

Ad:AD::Ae:AE.

Naturally, the demonstration holds, *mutatis mutandis*, also in the case of the hyperbola. And, in the case of the parabola, it *always* follows that if BD, CE, bd, ce are ordinates such that BD:DA::bd:da, CE:EA:: ce:ea, then also, AD:ad::AE:ae. The proof of this is similar to the one just given. For suppose Abc and ABC are parabolas with axes, Ae and AE, and *latera recta*, l and L, respectively (see fig. 12). As before, position Abc and ABC so that their vertices and axes coincide and let bd and ce be arbitrary ordinates to axis Ae. Join lines Ab and Ac and extend them to B and C on parabola ABC (there are such points by *Conica*, I.27) and drop perpendiculars BD and CE. Then, we have immediately (by *Elem.*VI.4),

bd:dA::BD:DA and ce:eA::CE:EA.

Again, one observes that B and C are the only points (on this side of the axis) from which ordinates may be drawn satisfying the proportion. Now, since also bd:BD::Ad:AD and ce:CE::Ae:AE, we have,

sq.bd:sq.BD::sq.Ad:sq.AD and sq.ce:sq.CE::sq.Ae:sq.AE

But, by *Conica*, I.11, sq.bd, sq.ce, sq.BD, sq.CE equal rect.l,Ad, rect.l,Ae, rect.L,AD, rect.L,AE, respectively, so that,

rect.l,Ad: rect.L,AD::sq.Ad:sq.AD and

rect.l,Ae: rect.L,AE::sq.Ae:sq.AE.

From this, it follows that, l:L::Ad:AD and l:L::Ae:AE, and, therefore,

Ad:AD::Ae:AE.

With these results, Apollonius might have written "Similar conic sections are such that if, when ordinates are drawn in them to fall on the axes, the ratios of the ordinates to the lengths they cut off from the axes from the vertex of the section are equal to one another, then the ratios to each other of the portions which the ordinates cut off from the axes are equal ratios." In this form, the definition would certainly be easier to use, for one could use the same construction used in the demonstrations just given to provide any number of ordinates having the same ratio to the lengths they cut off from the axis. Yet such considerations seem to have played no part at all in the way Apollonius framed his definition; if he knew these results, and it was doubtless within his power to know them, he ignored them and adopted a definition worded in such a way that the ratios between the ordinates and the lengths they cut off from the axes and the ratios of the portions which the ordinates cut off from the axes to each other appear to be taken up *independently*. Lest one think that this wording is in any way accidental, we should emphasize that it is not only in the definition that Apollonius treats these proportions independently but also when he *uses* the definition in the proofs of VI.11,12. Thus, for example, at the end of the first part of VI.12 in which Apollonius shows that the similarity of the "figures" of ellipses or hyperbolas implies the similarity of the ellipses or hyperbolas, Apollonius concludes, "Therefore the ratios of the perpendiculars BT, HΣ, EI [see fig. 13], to the amounts they cut off from the axis, namely, AK, AΘ, AZ, are [respectively] equal to the ratios of the perpendiculars ΔX, NΦ, ΛY, to the amounts they cut off from the axis, namely OΓ, ΞΓ, MΓ.

> *And* [emphasis added]* the ratios of the parts of AK (which is the axis of AB)[49] which the perpendiculars cut off, to the parts of ΓO (which is the axis of ΓΔ) which the perpendiculars cut off are equal.
> So the section AB is similar to ΓΔ.

[49] The parentheses are Toomer's. In his remarks about the editorial principles used in the translation, Toomer says that rounded parentheses such as these "... are used, not for additions by me, but either to make the structure of the thought clearer, or to enclose parts of mathematical expressions, for which we also employ { }" (*Apollonius* Conics *Books V to VII*, p. xci).

Likewise, at the start of the second part of VI.12 in which Apollonius shows that if the ellipses or hyperbolas are similar then so too will be their "figures," Apollonius writes,

> Then, since the two sections are similar, we draw in section AB some perpendiculars to the axis, BT, HΣ, EI, and in section ΓΔ perpendiculars ΔX, NΦ, ΛY, and let the ratios of these perpendiculars to the amounts they cut off from the two axes be equal [respectively], *and likewise* [emphasis added]* the ratios of the parts they [the perpendiculars] cut off from one of the two axes to the parts they [the corresponding perpendiculars] cut off from the other axis . . .

The reason, we think, why Apollonius's separating these two proportions appears as a problematic element in his definition of similar conic sections is that one assumes that, in his definition, Apollonius is looking for a criterion or condition for similarity (criteria and conditions he reserves for propositions) rather than for a precise *description* of figures having the same shape, a clear delineation of what one sees when one looks at such figures. Apollonius's definition reflects the sort of *simultaneous judgments* one makes when one looks at something that is spread out in space. It is as if Apollonius surveys the sections, as an artist might survey a building, say, a Greek temple, he is about to portray on paper; looking at the conic section, he says it is so high and so wide and it is so high and so wide here and here and here, that is, as a description the definition is a series of conjunctions rather than deductions. Thinking of Apollonius's definition in this way also explains Apollonius's apparent lack of concern as to the ease of its immediate application. The main difficulty in this regard is this: without linking the two sets of ratios in his definition, those of the ordinates to the lengths they cut off from the axes and those of the portions which the ordinates cut off from the axes to each other, how does one choose the ordinates so that the proportions will be satisfied? Suppose one has two ellipses and wants to check whether or not they are similar, how does one begin? One certainly cannot start with the segments cut off from the axis from the vertex with no further qualification, for it can easily happen, as in the following diagram (fig. 14), that BA:ba::AD:ad while it is not the case that CB:BA::cb:ba and ED:DA::ed:da.

The way Apollonius approaches the problem reveals further that in his thinking about the shape of conic sections he is looking right at them, as it were. Consider Apollonius's procedure in VI.12. Apollonius initially chooses a segment cut off from the axis from the

vertex in each ellipse or hyperbola having the same ratio to the axis, and then chooses other segments of the axis relative to this starting segment; having chosen these, he has naturally also chosen the ordinates to be compared. In Apollonius's words:

> We cut off from the axes segments AK, ΓO [see fig. 13], and let AK:AΠ = ΓO:ΓR.
>
> We cut AK arbitrarily at points Z, Θ, and cut ΓO into the same number of segments as AK and in the same ratios, at points M, Ξ.
>
> We draw from points Z, Θ, K, M, Ξ, O perpendiculars to the two axes BK, ΘH, ZE, OΔ, ΞN, MΛ, [and extend them to meet the sections again in T, Σ, I, X, Φ, Y].

In this construction, then, the axis is the reference for the position of the ordinates examined. But, we must stress, in this capacity it is an *internal* reference, a definite line segment in a definite position in the conic section, as the stairs might be for an artist's surveying a temple he is about to draw, to use the metaphor above; the axis is not a segment of an indefinitely extended and arbitrarily, though conveniently, placed measuring rod. Finding a similar internal reference for the parabola cannot be done quite so easily since the axis of the parabola is bounded only at one end. For parabolas, then, Apollonius takes initial segments cut of from the axes having the same ratio with the *latera recta.* This, in a way, gives the parabola what it sorely lacks, a finite axis which can be cut in definite ratios. Again, here is the procedure in Apollonius's words:

> We make [the parabolas']* *latera recta* AΠ, ΓP [fig. 13], and make (AK:AΠ) = (ΓO:ΓP).
>
> We cut AK at two points, Z, Θ, arbitrarily, and cut ΓO into the same number of segments, with the same ratio [as AK and its segments], at points M, Ξ.
>
> We draw from axes AK, ΓO, perpendiculars ZE, ΘH, KB, MΛ, NΞ, ΔO [and extend them to meet the sections again in I, Σ, T, Y, Φ, X].

It is clear that, for Apollonius, comparing these segments to a line such as the axis is preferable to one such as the *latus rectum.* For, in VI.12, Apollonius could as easily have taken segments AK, ΓO such that the ratio of AK to the *latus rectum* of the one ellipse or hyperbola equals the ratio of ΓO to the *latus rectum* of the other ellipse or hyperbola, instead of looking at the ratios of these segments to the axes; the proof would not change in any other essential way. The reason for Apollonius's preference should be evident from what was

said about diameters in the first chapter, namely, that the sectioning procedure producing the parabola, hyperbola, and ellipse produces *at the same time* their axes (when the cone is right) and ordinate direction. The axes and ordinate directions are, hence, things inseparably *in* conic sections. This recalls the comments above concerning Euclid's definition of similar circular segments. In them, we suggested that Euclid rejected a definition of similar circular segments in terms of the angle at the center of the circle subtended by the base of the segment because of a bias towards things *in* the figures being compared. Apollonius, so it seems, shares this same bias with Euclid. And indeed, to say that a definition of similarity ought to be in terms of things *in* the figures or, in other words, that the meaning of similarity is conveyed by such things, is to say, simply, that when one compares the shape of figures one looks *at* them and not *outside* of them.

In what we have said until now concerning similarity, there have been two major threads. The first is that there is no precise and well-defined framework within which one can define similarity; even ratio and proportion are not absolutely essential, as one learns upon finding, contrary to expectation, that similar circular segments are defined in Book III instead of in Book VI of the *Elements*, that is, *before* the discussion of ratio and proportion. Each type of figure seems to bring with it its own special definition of similarity.[50] This means that each kind of figure, whether it be a rectilineal figure, a solid, a segment of a circle, or a conic section, must be considered in its own peculiar context. The second is that, as with equality, Apollonius, in considering similarity, looks at the conic sections as *shapes* (rather than, for example, a representation of some abstract property, that is, a plane *locus*, in the modern sense). Accordingly, his definition is a visually motivated attempt to say what makes a conic section look like another, what makes them have the same shape. This is reflected in Apollonius's framing the definition in terms of the axis and ordinates rather than the "figure" on the axis, so that the focus of the definition is on things *in* the conic section, and

[50] In this connection, we remind the reader that among the definitions opening Book VI of the *Conica* there is, besides the definition of similar conic sections, a definition of similar right cones identical to Euclid's. So even restricting the discussion to the *Conica*, one can find different definitions of similarity for different kinds of figures.

in his presenting the crucial proportions as if independent, sacrificing, thus, the possibility of having a definition easy to use for the sake of maintaining the definition's descriptive character. These points, of course, overlap with what we have said about equality in the last section, which raises the question of the connection between equality and similarity. Spelling out this relationship is, we think, crucial for seeing how Apollonius understood similarity and what conceptual difficulties he faced regarding it. So let us now discuss this relationship.

Equality and Similarity

The obvious link between equality and similarity, from what has been said already, is the problem of sameness of shape. Indeed, much of what we have said, for example, about equality of parts of conic sections and the importance of this in understanding the shape of conic sections, we could only repeat regarding the *similarity* of parts of conic sections. But beyond this, how deep, really, is the link between equality and similarity?

The striking analogies among the propositions on equality and those on similarity must certainly impress themselves on any reader of Book VI. Proposition VI.2 says that ellipses and hyperbolas are equal whenever the "figures" on their axes are equal and similar, while VI.12 says that hyperbolas and ellipses are similar when the same "figures" are similar; proposition VI.3a says that hyperbolas and ellipses[51] are equal when the ordinates meet the diameter at equal angles and the "figures" constructed on the diameters are equal and similar, while VI.13 says that hyperbolas and ellipses are similar when the ordinates meet the diameter at equal angles and the "figures" constructed on the diameters are similar; VI.3b says parabolas are not equal to hyperbolas and ellipses are not equal to parabolas or hyperbolas, while VI.14 says "A parabola is not similar to a hyperbola nor to an ellipse," and VI.15, "A hyperbola is not similar to an ellipse"; VI.7 says that ". . . perpendiculars drawn from a

[51] VI.3a also refers to the parabola: "If there are [a number of] parabolas, and the ordinates falling on one of their diameters meet the diameters at equal angles [for each parabola], and their *latera recta* are equal, then the sections are equal . . ." Apollonius does need an analogous statement in VI.13 since he has proven in VI.11 that all parabolas are similar.

parabola or hyperbola to its axis, and extended to the other side, cut off from the section on both sides of the axis segments which, when one is applied to another, fit so as not to exceed or fall short of it, but do not fit on any other part of the section if placed on it," while VI.19 says that "When lines are drawn in parabola or hyperbola as perpendiculars to the axis, then the two segments cut off by each pair of perpendiculars on either side [of the axis] are similar and similar in position; but as for the other segments [in that section], they are dissimilar to them [the other pairs]"; VI.8 is the proposition about ellipses matching VI.7, while VI.20 is the proposition about ellipses matching VI.19; VI.9 says that "[In] equal sections, those parts of them at equal distances from their vertices will fit one on another, and those [parts] not at equal distances from their vertices will not fit one on another," while VI.21,22 say, essentially,[52] that in similar sections, those segments at proportional distances from their vertices are similar, and segments not at proportional distances are not similar; VI.10 says "[In] sections which are unequal, no part of one of them will fit on a part of another," while VI.23,24 say "[In] sections which are not similar, no segment of one of them is similar to a segment of another."

Yet, these analogies are not entirely consistent or complete. For example, in VI.3b Apollonius thinks it trivial that an ellipse cannot be equal to a parabola or hyperbola and goes on to prove only that a parabola cannot be equal to a hyperbola. He says, "As for the ellipse, it is obvious that it cannot be equal to any of the other sections, because it is bounded while they are unbounded." Pressed, one can expand the argument easily enough: suppose an ellipse and a hyperbola were equal, then they would coincide as would their

[52] The exact statement of VI.21 is "When lines are drawn in parabolas so as to be perpendiculars to the axes and to cut off from the axes in the direction of the vertices of the sections segments whose ratios to the *latera recta* in all the sections are equal, then the segments which those perpendiculars cut off from one of the sections are similar to the segments which the other perpendiculars cut off from the other sections, and their situation is similar, but they are not similar to any of the other segments that are taken from those sections," and that of VI.22, simply, "For similar hyperbolas and ellipses, the same properties hold as we proved hold for parabolas in the preceding proposition." The connection with VI.9 is justified when one considers that, in VI.9, what it means for segments to be "at the same distance" is that they be between perpendiculars to the axis at the same distance to the vertex or center as is suggested by the end of the proof (see Toomer, *Apollonius* Conics *Books V to VII*, p. 528, note 33).

axes. By I.8, one can find a point on the hyperbola such that a line drawn from the point to the axis cuts off from the axis a segment longer than the axis of the "equal" ellipse. Since the hyperbola coincides with the ellipse, this point coincides with a point on the ellipse. But this is absurd, for from a point on an ellipse one has drawn a line perpendicular to the axis that, nevertheless, meets the axis outside the ellipse. In fact, however, one reads Apollonius's remark almost without a second thought and he states it almost as if it were a truism. Indeed, we know from Proclus that in Geminus's classification of curves the division between bounded and unbounded curves was a basic one: "Some lines, he says, are finite and enclose a figure, like the circle, the perimeter of an ellipse, the cissoid, and many others; others are unlimited and can be extended indefinitely, like the straight line, the section of a right-angled or an obtuse-angled cone [*i.e.*, a parabola or hyperbola], and the conchoid."[53] Thus, one is somewhat surprised when, in VI.15, Apollonius does *not* think it trivial to prove that "A hyperbola is not *similar* [emphasis added] to an ellipse." Needless to say, boundedness plays no part in the proof of VI.15:

> Let there be a hyperbola, AB, and an ellipse, ΓΔ [fig. 15].
>
> Let their axes be [respectively] AK, ΓM, and let their transverse diameters be AE, ΓZ.
>
> Then, if these two sections are similar, then there are in the sections some perpendiculars, e.g. BN, ΘΞ, ΔO, ΛΠ, such that the ratios of these to the portions they cut off from the axes in both sections are [respectively] equal.
>
> Then we will prove as we proved in the preceding proposition that
>
> sq.ΘK:sq.BH = sq.ΛM:sq.ΔI
> = rect.EK,KA:rect. EH,HA
> = rect.ZM,MΓ:rect.ZI.IΓ.
>
> Therefore, rect.EK,KA:rect.EH,HA = rect.ZM,MΓ:rect.ZI,IΓ.
>
> And, when that is so and KA:AH = MΓ:ΓI, KE:EH = ZM:ZI: that is absurd.
>
> So section AB is not similar to section ΓΔ.

So, although Apollonius's approach to equality and his approach to similarity are equally geometrical and equally concerned with the problem of shape, it appears that the specific formulations of each make at least this aspect of shape, boundedness, an obvious category for the one but not for the other. Let us consider another example.

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[53] Proclus, *In Euclid.*, pp. 177–178 (Morrow's translation, p. 138).

Following the propositions (VI.23,24) proving that dissimilar conic sections contain no similar segments,[54] Apollonius proves that "It is not possible for a part of any of the three conic sections to be an arc of a circle" (VI.25). At first sight, this proposition appears to be in just the right place. For the general result stated at the start of VI.23 is broken into cases, VI.23 proving the result when the dissimilar conic sections are both hyperbolas or ellipses, and VI.24, when one section is a parabola and the other is a hyperbola or an ellipse. The next case one expects is that in which one section is a parabola, hyperbola, or ellipse and the other a circle. That Apollonius does not state VI.25 as a case (as he does in VI.24) but as a distinct proposition does not present a great problem in this respect: Apollonius often separates the circle from the other conic sections, as in II.28 (to take a random example), "If in a section of a cone *or circumference of a circle* [emphasis added] some straight line bisects two parallel straight lines, then it will be a diameter of the section." But in this sequence, VI.25 is peculiar in other ways. First, in VI.24, Apollonius can say ". . . make section AB a parabola and section ΓΔ a hyperbola or an ellipse," since he has already proven that these conic sections are all dissimilar (VI.14,15). He has *not* proven, however, that a circle and the other conic sections are dissimilar, and we stress here (although it is, perhaps, unnecessary to say) that *nowhere* in the *Conica* does Apollonius ever see the circle as a kind of ellipse, so that, for Apollonius, proving an ellipse and the other conic sections dissimilar proves nothing about the circle. Second, and more important, VI.25 is not immediately a proposition about similarity but about *equality*. In fact, one could easily see it as a mere corollary to VI.10, "[In] sections which are unequal, no part of one of them will fit on a part of another." So what is it doing in the section of Book VI on *similarity*? Or, put another way, why does Apollonius not prove VI.25 after VI.10 and then prove its complement, that "it is not possible for a part of any of the three conic sections to be similar to an arc of a circle" after VI.24?

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[54] Similar segments are defined by Apollonius as ". . . those in which the angles formed between their bases and their diameters are equal, and for which, an equal number of lines having been drawn in each of them parallel to their base, the ratios of these lines (and also the ratio of [each] base) to the lengths which they cut off from the diameter from the vertex of the section are equal for every segment; similarly the ratio of the part cut off from the diameter of one to the [corresponding] part cut off from the diameter of the other [must be equal in all cases]."

We can make a start by observing that the missing proposition, that no part of a parabola, hyperbola, or ellipse is *similar* to a part of a circle, follows from VI.25. For suppose a segment of a circle, ABC, were similar to a segment, abc, of some other conic section.[55] And suppose the base of the segment of the circle cuts the diameter drawn perpendicularly to it at point E and that of the base of the segment in the other section cuts the diameter similarly drawn at point e (see fig. 16). Then, since ABC is a segment of a circle, angle AEB is right, and, therefore, also angle aeb is right. Again, since the segments are assumed similar, we have BE:EA::be:ea. Draw a circle through points a, b, c. Then, by VI.25, the arc abc of this circle does not coincide with segment abc of the section. Let d be a point on the circle arc which is not on the segment of the section, and draw dfg parallel to bec. Draw DG parallel to BEC and such that AG:GE::ag:ge. Now, circle segment ABC is similar to segment abc since BE:EA::be:ea and angle AEB equals aeb, and, therefore, dg:ag::DG:AG. But by the similarity of the segment ABC and the segment abc of the section, fg:ag::DG:AG. Therefore, dg equals fg, which is absurd.

We think it is fair to say that while this demonstration is not difficult, neither is it as immediate as some other demonstrations Apollonius chooses to include in the *Conica*. If we assume there is no lacuna here, then this, as well as the placement of VI.25 after VI.24 instead of after, say, VI.10, implies that, rather than leaving out an argument, Apollonius simply thought that the argument given was completely sufficient for the conclusion regarding similarity. In other words, *in this one instance*, one has a proposition which assumes that if a conic section or segment of a conic section cannot be equal to another conic section or segment of a conic section, then it cannot be similar either. And, of course, it is *only* in this instance that Apollonius makes such an assumption, as other pairs such as VI.3b and VI.13,14, VI.10 and VI.23,24 make quite clear. Apollonius *does* take the statement, that if sections or parts of sections can be equal then they can be similar, as obvious and generally true, though, perhaps, not so much so that there is no need to point it out. Consider, for example VI.19,[56] which states that "When lines are drawn in

[55] See the previous note.

[56] Other examples are VI.20, which is the analogue to VI.19 for the case of the

parabola or hyperbola as perpendiculars to the axis, then the two segments cut off by each pair of perpendiculars on either side [of the axis] are similar and similar in position; but as for the other segments [in that section], they are dissimilar to them [the other pairs]." Having set ΒΓ and ΘΚ as the segments cut off by the perpendiculars and ΔΕ and ΘΚ as segments not cut off by the same pair of perpendiculars, Apollonius begins the proof of VI.19 by saying, "As for [the statement] that segments ΒΓ and ΘΚ are similar, that is obvious, because each of them will fit on the other, as is proven in Prop.7 of this Book." One does not have to search far for the reason why this statement is obvious: since the segments coincide, the bases coincide, and lines drawn in each segment parallel to the base and cutting off *equal* segments from the diameter (measured from the vertex) form equal angles with the diameter and have the same ratio with those segments cut off from the diameter. It is important to observe that the similarity of the segments follows from their equality only because, being equal, they are also similar *by Apollonius's definition of the term*, and not, as one might expect, because being equal they fit exactly one on the other and, therefore, have the same shape, *i.e.*, they are *similar*. One must understand that, although both "equality" and "similarity" are directed towards the problem of sameness of shape *they are at the same time fundamentally different approaches to this problem*. So, for Apollonius, while similarity of conic sections (or segments of conic sections) follows from their equality, and that only because the definition is fulfilled trivially when conic sections coincide, it does not follow immediately, except in the very special case

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ellipse, and VI.16, which states that "Opposite sections are similar and equal." It is worth remarking that the latter is exceptional in that the equality and similarity are proven at once. If "similar and equal" was the phrase ἴσος τε καὶ ὅμοιος in the Greek original, with Thābit changing the word order for some reason, then this would be the only instance in Book VI in which Apollonius uses the phrase in direct reference to conic sections (he does use it, presumably, in reference to other figures, as in VI.2, where he uses it regarding the rectangular "figures" belonging to hyperbolas and ellipses). We rather think that Thābit did not change the word order and the proposition was as he translated it. For the phrase ἴσος τε καὶ ὅμοιος is usually used in the way Apollonius uses the word "equal" alone; Apollonius might have used the phrase ἴσος τε καὶ ὅμοιος for every one of the first ten propositions and, thus, it would be strange if all of a sudden he used it here and here only. In VI.16, then, Apollonius is showing the two things *separately*, albeit in one proposition, that is, that the opposite sections are "equal" *and* "similar" according to his definition of the terms.

of the circle, that the possibility of being similar implies the possibility of being equal; equality is not a mere special case of similarity.

Apollonius's understanding of equality and similarity as truly distinct approaches to the problem of the shape of conic sections means that Zeuthen cannot be right when he says, referring to the definition of similar conic sections, "Diese Definition ist von allgemeiner Natur, wenn sie auch unmittelbar nur an Kegelschnitte angeschlossen ist, die auf eine Axe und eine Scheitelpunktstangente als Koordinatenaxen bezogen sind. Sie ist nämlich nur eine specielle, für den Augenblick verwendbare Form von folgender Definition—welche die Alten allerdings nicht ausgesprochen haben—: Kurven heißen ähnlich, wenn sie auf ein rechtwinkliges Koordinatensystem sich so beziehen lassen, daß die beiden Koordinaten entsprechender Punkte in einem und demselben konstanten Verhältnis stehen."[57] As we have said before, the coordinate systems of modern analytic geometry shift the focus from particular curves and their particular geometrical origins, which we have argued is the ancient approach, to the abstract metrical relations within a particular space, the coordinate plane.[58] From this latter point of view, which is evidently Zeuthen's starting point, similarity is not a matter of comparing two shapes but of comparing the coordinate systems in which they are found. Hence, if a curve is given by the coordinates (x,y) of the points in the plane through which the curve passes, and another curve (in a similar position with respect to the coordinate system) by the coordinates (x',y'), then the two curves will be similar if the coordinates in the two systems are

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[57] Zeuthen, *Die Lehre*, p. 387.

[58] In Archimedes's *On Conoids and Spheroids*, the diagram accompanying proposition 6 might lead one to think that, indeed, there was in the ancient world a conception of similarity as a kind of expansion of dimensions, a conception closely related to the algebraic conception suggested by Zeuthen. A closer look, naturally, shows that the diagram in this respect is deceiving. The proposition states that ". . . the areas contained by sections of an acute angled cone [*i.e.*, an ellipse] have the same ratio as those contained by the rectangles having as their sides the diameters [*i.e.*, axes] of the section" (our translation). Cutting the sections from a single cone by a plane perpendicular to the side of the cone, the ellipses in the proposition would be similar (see *Conica* VI.27), and, indeed, in the *corollary* following proposition 6, Archimedes does relate the result to similar ellipses. However, the proposition is concerned only with the relationship between ellipses and the rectangles containing them and as such one should not see in the rectangles anything more than rectangles. Moreover, the proposition does not suppose that the ellipses (or the rectangles) be similar and it does not have to; the proposition would be equally true if the ellipses were not similar.

related by the equation x/x' = y/y' or, being that we are already in modern territory, if there is a single number k such that x = kx' and y = ky'. Equality is, then, simply the case when k = 1. Moreover, if two curves are equal, so that the coordinates of their points satisfy the equations x = x' and y = y', then if there is another curve whose coordinates (x",y") satisfy x" = kx' and y" = ky', this curve will be similar to the first two. In this algebraic framework, the distinction between similarity and equality has little force; if shapes can be equal, then they can, by the change of coordinates above, be similar and if they cannot be equal, then they cannot be similar. Such a framework, even if unspoken, would make pairs such as VI.12 and VI.2 quite unnecessary; Apollonius could simply say ellipses are similar when the figures on the axis are similar, and equal, if he felt the need to say so, when equal and similar.

The theme of the first two sections in this chapter was equality and similarity as a study of shape. The present section has stressed that, at the same time, equality and similarity represent, for Apollonius, two radically different approaches to the study of shape. Hence, on the one hand the propositions concerning equality have analogies among the propositions concerning similarity, while on the other hand, the analogies are incomplete. Of the two, it is clear that, conceptually, similarity is the more complex and more problematic; equality does not present the same difficulty since, no matter how one looks at shape, it is obvious that two objects will have the *same* shape if they coincide exactly. Finding what makes geometrical objects such as conic sections have the same shape when they *cannot* be made to coincide exactly was the challenge Apollonius faced in his thinking about similarity. A sign that the similarity of conic sections was indeed a challenge, *conceptually* (although not necessarily *technically*), for Apollonius is the rigor with which he applied the definition when he finally arrived at it. Thus, when he considers whether an ellipse can be *equal* to a hyperbola, he dismisses the possibility *tout court* saying "it is *obvious* [emphasis added]* that it cannot be equal to any of the other sections, because it is bounded while they are unbounded." However, when he considers whether an ellipse can be *similar* to a hyperbola, he *proves* that it cannot, strictly adhering to his definition; he does *not* say that it is *obvious* that an ellipse cannot be similar to a hyperbola. Whether Apollonius ever genuinely weighed the possibility of a bounded curve being similar to an unbounded one is not the point: what is, is that when Apollonius considered

questions of similarity, he took no chances, he used the definition uncompromisingly and let no notions such as "boundedness" enter the argument. The final point in this section was that, once one understands that Apollonius's thinking about equality and similarity was on two distinct tracks, one understands that it is very unlikely he thought of similarity in terms of proportionally expanding coordinates, since this puts similarity and equality in one continuum as it were. To gain some understanding of what importance similarity and equality held for Apollonius, we must now consider the last part of Book VI, the constructions.

The Constructions

Book VI, and especially its end, makes it clear that although Apollonius seemingly abandons the cone after Book I, in fact, the cone is ever in the background of the *Conica.* Propositions VI.26,27 are the first propositions in which the cone comes again explicitly into the foreground. Proposition VI.26 states that "If cones are cut on one side [of their axes] by parallel planes from the class of planes which, when extended on the side of the vertex of the cone, subtend its exterior angle,[59] then the hyperbolas generated [by these planes] are similar but not equal," while VI.27 states the analogous result for the ellipse.[60] To see the status of the cone in these propositions, look at the proof of, say, VI.26:

> Let there be a cone, ΑΒΓ [see fig. 17[61]] and let it be cut by two parallel planes, and let their common sections with the base [of the cone] be the lines ΘΜ and ΚΝ.

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[59] Regarding the phrase "... which, when extended on the side of the vertex of the cone, subtend its exterior angle ...," Toomer remarks, justifiably, "*I.e.* which cut the generator of the cone when it is extended, and thus (by I.12) generate a hyperbola" (*Apollonius* Conics *Books V to VII*, p. 588, note 145).

[60] Heath notes "A similar result holds for the *parabola*, since by Prop. 1 [in Heath], PL:PA [*i.e.*, the ratio between the parameter and the segment from the vertex of the cone to the vertex of the parabola] is a constant ratio. Therefore two parallel parabolic sections have different parameters" (Heath, *Apollonius of Perga: Treatise on Conic Sections*, p. 208).

[61] This is the diagram (Toomer's diagram 6.26*) that Halley uses following a scribal emendation of the Oxford manuscript of the Banū Mūsā text. In the original diagram, the hyperbolas ΝΔΚ and ΜΖΘ are switched, so that the diagram would not be consistent with the end of the proof where it is said that rect.ΟΔ,ΔΞ is *greater* than rect.ΕΖ,ΖΙ (see Toomer, *Apollonius* Conics *Books V to VII*, p. 589, note 149 for further details).

We draw from the center of the base of the cone a perpendicular to these lines BΛHΓ.

Let the cone be cut by [another] plane passing through line BΓ and the axis of the cone, and let this plane cut the surface of the cone in lines AB, AΓ.

Let the common sections of this plane with the two parallel planes be lines ΔΛ, ZH: we extend them [ΔΛ, ZH] to [meet ΓA extended in] O, E [respectively].

Then I say that the section ΘZM is similar to section KΔN, but not equal to it.

Proof of that:

We draw from point A a line parallel to lines ΔΛ, ZH, namely AΠ.

We make (OΔ:ΔX) = sq.AΠ:rect.BΠ,ΠΓ and also EZ:ZI = sq.AΠ: rect.BΠ,ΠΓ.

Then, since line BΛ is perpendicular to line KN, the lines drawn in hyperbola KΔN to line ΔΛ parallel to line KN are equal in square to the rectangles applied to ΔΞ (which is the *latus rectum*), and exceeding it by a rectangle similar to the rectangle contained by lines OΔ and ΔΞ, as proven in Prop.12 of Book I.

Similarly too the lines drawn in hyperbola ΘZM to line ZH parallel to line ΘM are equal in square to the rectangles applied to ZI (which is the *latus rectum*) and exceeding it by a rectangle similar to the rectangle contained by lines EZ and ZI.

And the angles formed by line KN with line ΔΛ are equal to the angles formed by line ΘM with line ZH, because they [KN, ΔΛ] are parallel to them [ΘM, ZH respectively].

So the sections are similar, as is proven in Prop.12 of this Book.

And (OΔ.ΔΞ) is greater than [*sic*] (EZ.ZI).

So segments [*sic*] ΘZM, KΔN are unequal, because of what is proven in Prop.2 of this Book.

One sees here that the four intervening books between Books I and V have changed nothing in Apollonius's initial conception of conic sections as *sections* of a cone; the way in which the cone is used in the proposition to define the section, the *latus rectum*, and the "figure" is virtually identical to the way it is used in Book I. In fact, had Apollonius given the definition of similarity in Book I, one could easily see propositions VI.26,27 following I.11–13 to suggest an analogue to the first part of proposition I.4, namely, that "If either one of the vertically opposite surfaces is cut by some plane parallel to the circle along which the straight line generating the surface is moved, the plane cut off within the surface will be a circle having its center on the axis . . ." These propositions are in the later book, Book VI, rather than the earlier book, Book I, only because the notion of similarity had first to be given, understood, and accepted

before the propositions would make sense; other than that, they might as well have been in Book I.

As for their place within Book VI, propositions VI.26,27 serve as *lemmas* for the constructions, VI.28–33; they do not themselves belong among the constructions since they are not constructions, nor, indeed, do they belong among the propositions on similarity since, if they did, they would surely appear *before* VI.17, that is, before the propositions on similarity of *segments* and among the propositions concerning *whole sections*. Now, the constructions concern only *equal* conic sections: VI.28–30 show how one can construct a (right) cone containing a section equal to a given section, while VI.31–33 show how one can construct a cone similar to a given (right) cone and containing a given conic section. Propositions VI.26,27 make constructing a cone containing a section *similar* to a given section unnecessary, since having constructed a cone containing a section equal to a given conic section, one has at the same time constructed a cone containing a section similar to the given conic section (recall, Apollonius does accept that if sections are equal they are similar), and parallel sections will then provide, by VI.26,27,[62] sections similar but not equal to the given section. So, with VI.26,27, one need only concern oneself with equality in the constructions, and, given their position in the book, this appears to be their chief function. Let us, then, turn to the constructions themselves.

Looked at in the context of the whole of the *Conica*, what is most striking about the constructions at the end of Book VI is their similarity to those at the end of Book I,[63] both in their apparent content and in the methods used in their demonstration. As an example of the latter, consider I.54 in which, given two perpendicular lines, Apollonius shows how to find a hyperbola in the plane through these lines and having one of the lines as its transverse diameter and the other as its upright side. Thus, suppose AB, BΓ are the given perpendicular lines and AB has been extended to Δ. Then, it is required to find in the plane, ABΓ, a hyperbola whose diameter is ABΔ, whose vertex is B, and whose upright side is BΓ. In the first case (I.54; the second case is I.55), the ordinate direction is at right angles to the

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[62] See note 59, *supra*.

[63] Indeed, this similarity is mentioned by Zeuthen (*Die Lehre*, p. 384), Heath (*A History of Greek Mathematics*, II,168), and Toomer (*Apollonius* Conics *Books V to VII*, p. lxii).

diameter, *i.e.*, the diameter is the axis. As in the other constructions at the end of Book I, I.52 and I.56, Apollonius proceeds by finding a right cone cut by the plane of reference (that formed by AB, BΓ, in the present case) so that the section formed is the section required. Since the cone is right, it suffices to find an axial triangle of the cone given in position. In I.54, Apollonius uses an auxiliary circle to find such an axial triangle, a device which, as we shall see, returns in VI.32. We shall paraphrase his procedure. First, draw a circle AEBZ (see fig. 18[64]) about AB in a plane perpendiular to the plane of reference and such that the ratio of the part of the diameter in sector AEB to that in AZB is not greater than the ratio AB:BΓ.[65] Divide AEB into two equal parts at E and draw EK perpendicular to AB. Now, either AB:BΓ::EK:KΛ or AB:BΓ > EK:KΛ. If the latter, divide KΛ at M such that AB:BΓ::EK:KM. Draw MZ parallel to AB and join AZ, EZ, ZB. Draw BΞ parallel to ZE. Since angle AZE equals angle EZB (recall, arcs AE and EB are equal by construction), angle AZE equals AΞB, ΞBZ equals EZB, it follows that angle ZΞB equals angle ZBΞ. Hence, triangle BZΞ is isosceles and, with that, one can conceive a cone having this as its axial triangle, that is, a cone whose vertex is Z and whose base is the circle with diameter BΞ and in the plane perpendicular to BZΞ. In VI.32, whose task it is "... to show how to construct a right cone, similar to a given right cone, containing a given hyperbola," Apollonius again uses an auxiliary circle such as that in I.54 to find an axial triangle of the required cone. Again, we shall only paraphrase the relevant parts of VI.32. Suppose a hyperbola is given having axis AΛ (see fig. 19), transverse diameter AN, and whose figure is bounded by NA and AΔ. Also, suppose a cone is given whose axial triangle, EZK, is such that sq.EH:sq.ZH is not greater than AN:AΔ and whose external angle at E is the angle ΨEZ. Now, draw a segment of a circle on AN, in the plane perpendicular to plane containing the hyperbola, and such that the segment NΘA contains an angle equal to angle ΨEZ. Complete the circle and bisect the arc NΘA at point Θ. Draw ΘΞ perpendicular to AN and extend ΘΞ to Σ. In the case where sq.EH:sq.ZH::NA:AΔ, one produces NΘ beyond point Θ and draws AM parallel to ΘΞ to meet NΘ at point M. It is then easy

[64] We have included here only the relevant parts of the diagram accompaying I.54 to bring out the similarity with VI.32.

to show that triangle AΘM is isosceles and similar to ZEK. The required cone, accordingly, will be that having its vertex at Θ, and its base the circle about diameter AM and in the plane perpendicular to ΘΑΛ. In the case where sq.EH:sq.ZH < ΝΑ:ΑΔ, Apollonius first shows that that this condition is equivalent to the condition ΣΞ:ΞΘ < ΝΑ:ΑΔ and then sets ΣΞ:ΞX::ΝΑ:ΑΔ. From here, Apollonius procedes precisely as in I.54, he draws IXT parallel to NA, joins IN, IΣ, IA, and draws AO parallel to IΣ. Just as before, triangle OIA is seen to be isosceles and similar to triangle ZEK, so that, again, the required cone will be that having its vertex at I and its base the circle about diameter AO and in the plane perpendicular to ΘΑΛ.

With no real technical innovations to be found in the constructions at the end of Book VI, one must assume that, if there is a difference between these constructions and those in Book I, it lies in their content. And here we should say that it is highly unlikely that there is *no* essential difference between these two sets of constructions, that the constructions in Book VI are a mere elaboration of those in Book I,[66] for in the introductory letter to Book VI Apollonius places some importance on these constructions, telling Attalus, "In this Book there is also how to find a section in a given right cone equal to a given section, and how to find a right cone, containing a given conic section, similar to a given cone."[67] Needless to say, Apollonius does not mention the constructions in Book I in the introduction to that Book. This state of affairs would be difficult to understand if Book VI added nothing fundamental to the real achievement that was in Book I. So what is new in the constructions in Book VI? To answer this, let us compare the tasks in I.52 and VI.31. The choice is made for the sake of simplicity alone; other corresponding pairs would have done quite as well.

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[65] Eutocius shows how this may be done.

[66] Compare Zeuthen, *Die Lehre*, p. 384.

[67] One might object that Apollonius does not mention the constructions in Book VI when he reviews the contents of the various books of the *Conica* in the introductory letter to Book I. However, in that letter Apollonius spells out the contents of the elementary books in considerably more detail than of the later books. Books V, VI, VII, and VIII are summarized in one sentence: ". . . there is one [book] dealing more fully with maxima and minima [Book V], and one with equal and similar sections of a cone[Book VI], and one with limiting theorems [Book VII], and one with determinate conic problems [Book VIII]." Indeed, Apollonius seems to be giving us in Book I more broad topics of the later books than genuine descriptions of their contents.

Proposition I.52 proposes the problem "Given a straight line in a plane bounded at one point, to find in the plane the section of a cone called parabola, whose diameter is the given straight line, and whose vertex is the end of the straight line, and where whatever straight line is dropped from the section to the diameter at a given angle, will equal in square the rectangle contained by the straight line cut off by it from the vertex of the section and by some other given straight line." Accordingly, Apollonius sets out the problem as follows: "Let there be the straight line AB given in position and bounded at the point A, and another straight line CD given in magnitude, and first let the given angle be a right angle; it is required then to find a parabola in the plane of reference whose diameter is the straight line AB and whose vertex is the point A, and whose upright side is the straight line CD, and where the straight lines dropped ordinatewise will be dropped at a right angle, that is so that AB is the axis." So, given here is a point and a line in position, a plane (implicitly), and another line segment in magnitude. What has to be found is a parabola having the line and point as its axis and vertex, respectively, and the line segment as its upright side (or *latus rectum*). How does this compare to VI.31.

Proposition VI.31 presents the problem, ". . . how to find a right cone containing a given parabola and similar to a given right cone." In setting out the problem, Apollonius says, accordingly, "Let the parabola be ΒΑΓ, with axis ΑΛ, and the *latus rectum* for that section, ΑΔ, and the given cone ΕΖΚ, with the triangle passing through its axis ΕΖΚ." Here, then, what is given is a parabola together with its axis and *latus rectum*, and a right cone with, Apollonius emphasizes, its axial triangle. Let us look first at the second given, the right cone and its axial triangle. These two are given together, and labeled in exactly the same way, since the treatment of the right cone is, in practice, largely a treatment of its axial triangle. This is so chiefly because the right cone may be built by means of the axial triangle, as indeed it is, both here and in I.52. Moreover, when Apollonius concludes, in the demonstration of VI.31, that he has constructed the *required* cone, the reason he gives is that the axial triangles of the given and constructed cones are similar. In this connection, we remind the reader that Apollonius defines similar right cones as ". . . those for which the ratios of their axes to the diameters of their bases are equal," and, of course, this is equivalent to saying that similar right cones are those for which their axial triangles are similar, the axial triangles of right cones being isosceles. That said, one

easily sees that, Apollonius's defining similar right cones in Book VI notwithstanding, it would be far fetched to say that the subject of similar right cones, *per se*, is in any way a central issue in this Book; his definition of similar cones is no different from Euclid's definition[68] and his treatment of the cone here involves nothing new or particularly Apollonian, not even his own general definition of the cone given in Book I. So, without denying that the right cone being given in form *is* a difference between VI.31 and I.52, indeed, one that changes the details of the demonstration, we find it hard to accept that it is for *this* difference that Apollonius stresses constructions in his letter to Attalus. This leaves, then, the next noteworthy difference between the two constructions, namely, that in I.52 what is given is the vertex, axis, and upright side of a parabola to be found, while in VI.31 what is given is *the parabola itself* together with its axis and upright side. In fact, in *all* of the constructions, VI.28–33, there is a conic section said to be *given*. This is conspicuous since, except for the series of problems in Book II.44–51,[69] Apollonius rarely says that a conic section is *given*. Assuming that this *is* the crucial difference, one surely asks, what does the *givenness* of a section have to do with Book VI? To answer this, consider the first part of VI.31.[70]

> We pass through line AΛ a plane, ΘΛ [see fig. 20], at right angles to the plane in which lies section BAΓ, and draw in that plane [ΘΛ] line AM, which we make form, together with line AΛ, an angle equal to < EZK.
>
> We make (ΔA:AM) = (KZ:ZE), and construct on AM a triangle similar to triangle EZK, namely AΘM, and draw lines ΘA, ΘM from

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[68] Euclid defines similar cones and cylinders as ". . . those in which the axes and the diameters of the bases are proportional" (*Elem.*XI, def.24).

[69] It is of some interest that of the propositions used in Book VI from other books in the *Conica*, almost all are from Book I. Those which are not are II.7, II.28, and II.48. Of these, II.7 is used once, in VI.6; II.28 is used three times, in VI.19,20,25; II.48 is used once, in VI.25. The first problem in the series, II.44–51, is really proposition II.28 in problem form. There is, then, a certain correspondence between the propositions in Book II where conic sections are *given* and the propositions Apollonius uses from that book in Book VI.

[70] The second part of VI.31 asserts that no other right cone similar to the given cone contains the given section. The proof uses no proposition from Book VI. We should say, however, that this part of the proof is interesting, though not of immediate importance in the present discussion, for two reasons: 1) it goes beyond what Apollonius proposes in the enunciation, and 2) it marks another difference between I.52 and VI.31, namely, that in the former the construction implies that an infinite number of cones satisfies the conditions of the problem, while in the latter only one cone is possible.

points A and M, and construct a cone with vertex Θ and base the circle constructed on line AM, which is its diameter, and perpendicular to plane AΘM.

Then < MAΛ = < EZK.

But < EZK = < ΘMA.

Therefore, < MAΛ = < ΘMA.

Therefore line AΛ is parallel to line ΘM, which is a side of the triangle passing through the axis [of the cone]. So the plane in which lies the given section generates in the cone a parabola.

Up to this point, Apollonius's procedure differs somewhat from that used in I.52, mostly because of the additional requirement that the cone constructed here be similar to a given cone. However, there is nothing, so far, that could not have been done in Book I, and, in rough outline, the strategy here is the same as in I.52: find an isosceles triangle in a plane perpendicular to the plane of reference and having one side parallel to the axis of the section (so that the section will be a parabola), then conceive a cone having this as its axial triangle. Having carried out the actual construction, Apollonius finishes the first part of the proof thus:

And ΔA:AM = KZ:ZE = AM:MΘ.

Therefore AΔ:AM = AM:AΘ, because AΘ = MΘ.

Therefore sq.AM:sq.AΘ = AΔ:AΘ.

But sq.AΘ = AΘ.ΘM. Therefore sq.MA:(AΘ.ΘM) = ΔA:AΘ.

Therefore the *latus rectum* to the section generated in the cone is line ΔA. But it [ΔA] is also the *latus rectum* to section BAΓ. And parabolas for which the *latera recta* are equal are [themselves] equal, as is proven in Prop.1 of this Book.

Therefore section BAΓ is placed in the cone which we constructed, and the cone which we constructed is similar to cone EZK, because ΔEZK is similar to ΔAΘM.

It is fitting that, in the penultimate sentence, Apollonius (or Thâbit b. Qurra, his Arabic translator)[71] adds, ". . . as . . . proven in Prop.1

[71] Apollonius rarely cites previous propositions in the earlier books for which we have the Greek text, and so it may well be that this is an addition by the translator. Some evidence for this can be found in the preface to the Arabic translation by the Banū Mūsā, where they write, "And he [Ahmad b. Mūsā]* did something for this treatise which is of great use in facilitating the comprehension of it for anyone who wants to read it—something which Apollonius did not do when he composed it, nor Eutocius when he restored it. Namely he examined every premise which one needs for the proof of each of the propositions, mentioned it explicitly in the place where one needs it, and described its position in the treatise" (in Toomer, *Apollonius* Conics *Books V to VII*, p. 628; see also Toomer, p. xviii).

of this Book," for *only* here does Apollonius use anything proven in Book VI. So in this sentence we have the link between the subject matter of Book VI and the *givenness* of conic sections. For recall, the proposition requires finding ". . . a right cone containing a given parabola . . .," yet, what is found, in fact, is a right cone containing a parabola *equal* to a given parabola. For a moment then, it appears that there are *two* parabolas, the given parabola and the parabola equal to it. However, as if to prevent that thought, Apollonius immediately returns our attention to *one* section and concludes, "Therefore section ΒΑΓ [the given section] is placed in the cone which we constructed . . ." Evidently, then, to be a given section in a cone means that the cone contains a section that one can prove equal to the section said to be "given," or, put a little more boldly, a conic section is *given* precisely when it is possible to find another conic section *equal* to it. Put this way, one is reminded of Euclid's definition in the *Data*, of a geometric entity, a plane, line, or angle, being given in magnitude as one for which an equal can be found.[72] It would be going too far, of course, to say that Book VI was, in this respect, a book like the *Data*: for one thing, unlike in the *Data*, Apollonius never specifically defines a "given" conic section in the book. On the other hand, from what has been said, it is plain that such things were not far from Apollonius's thoughts. Moreover, that Apollonius should think about what makes a conic given, is, at least, consistent with other writings by him. For example, Marinus tells us that ". . . among those who made it their aim to define the datum (τὸ δεδομένον) more simply and with a single *differentia*, some called it simply *assigned* (τεταγμένον), such as Apollonius in his book *On Vergings* and in his *General Treatise*, others the *known* (γνώριμον), such as Diodorus."[73] Indeed, a scholium to the *Data* attributes definitions 13–15 in that book to Apollonius.[74]

It is, without a doubt, difficult for modern readers to accept that, on the basis of what has been shown in Book I, the parabola in I.52 is, for Apollonius, still not completely given. This is not to say

[72] Euclid, *Data*, definition 1. The full statement is, "Δεδομένα τῷ μεγέθει λέγεται χωρία τε καὶ γραμμαὶ καὶ γωνίαι, οἷς δυνάμεθα ἴσα πορίσασθαι." See Taisbak, "Euclid's Data" (The Bar-Hillel Colloquim, March 1995), pp. 5f, for an enlightening discussion of the use of πορίσασθαι here.

[73] Marinus, *in Eucl. Dat.*, Eucl. ed. Heiberg vi. 234. 13–17 (translation by Ivor Thomas in *Greek Mathematics*, II, 349–351).

[74] Cf. Heiberg, *Euclidis Data*, p. 5, note 2.

that I.52 is incomplete or blemished in any way; it sets out a problem and solves it. In I.52, Apollonius must find a parabola having a certain diameter and upright side, and this he does, always with one eye on the geometric source of its generation, exactly in accordance with what has been developed in Book I. But, as we have noted elsewhere, the construction in I.52 is such that any of an infinite number of cones provide the required parabola. If one were to carry out the construction once with one cone and once with another, would the parabolas be the same? It is certainly true that both would have the same diameter and upright side and, therefore, the squares on ordinates cutting off equal segments from the vertex of each parabola would be equal. In a *technical* sense this, or the analogous συμπτώματα for the other conic sections, is really all one needs to know if one wants to use a conic section in a proposition. And, needless to say, there is no lack of examples in which conic sections are used to solve certain problems,[75] problems which Pappus calls "solid (στερεά);"[76] Apollonius himself uses conic sections in the course of propositions, in V.51 for example. Viewed in this technical sense, the question of whether two parabolas, for which the squares on the ordinates are equal to the same rectangles, are the same seems rather trivial and, thus, VI.1 supererogatory. One is led away from such conclusions, however, when one realizes that in asking whether the two parabolas are the same, one is as much asking about the nature of a *single parabola* as about the relationship of two; to be able to say when it is that two parabolas occupy exactly the same place is to be able to say something about the place of one parabola alone. Taken in isolation, of course, VI.1 would not give an unambiguous answer as to whether Apollonius means to focus on the individual parabola or the relationship among several parabolas, for, after all does he not speak there and afterwords of *two* parabolas, *two* hyperbolas, *two* ellipses? But the emphasis in each construction, especially in the second set, VI.31–33, on a single "given" conic section gives persuasive evidence that throughout Book VI, Apollonius was, in fact, aiming towards individual conic sections and their nature as independent objects.

Let us now summarize and tie together some of the things we have been saying. In I.52, the attentive reader learns that many

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[75] Cf. Knorr, *The Ancient Tradition of Geometric Problems, passim.*
[76] Pappus, *Coll.* iv. 36, ed. Hultch 270.

cones will produce a parabola having a given axis and upright side. Each instance is an inseparable pair, cone plus conic section. Naturally, the sequence in producing such a pair is 1) two lines given in a plane, one in position only, one in magnitude, 2) a cone produced, and with that, 3) a conic section. This is the sequence one should expect in Book I, because Book I focuses on the *generation* of conic sections, as Apollonius stresses both in the introduction to Book I and also in the introduction to Book V. The definitions in Book I also reflect Apollonius's concern with the generation of conic sections in that they describe conic surfaces, cones, and general properties of curves, *but not conic sections*; conic sections are not a starting point in Book I, cones are. In Book VI, the order is reversed. The definitions first concern conic sections, then segments of conic sections, then sections contained by cones. Consistent with this, VI.31 begins with a conic section, its axis, and upright side and then builds a specific cone containing it. In other words, in VI.31 Apollonius shows how one can go from a parabola to a specific origin in a cone. VI.31 is, in this sense, a better candidate as a converse to I.11 than is I.52. For in I.52, one must have the cone first before one can begin speaking about a parabola. But the effort in Book VI has allowed Apollonius, heuristically at least, to *detach* the parabola from the cone, and take it as "given." And is this not what it *means* that something can be "given," namely that an object can be a starting point either with respect to its position or its size or its shape? In other places in the *Conica*, when Apollonius says "Let ABC be a parabola," he may not mention the cone, but one must imagine that ABC has been produced in some cone in the way prescribed by I.11. In Book VI, where sections start being *given*, sections are once again set in the company of the cone, for it is only thus that we can gauge whether they can stand alone.

Conclusion

The first three sections of this chapter showed that much of Book VI centered around the problem of shape. The question, naturally, is how to say precisely what is the shape of a parabola, of a hyperbola, of an ellipse. It is not easy to pose this question meaningfully; what, after all, does one actually *mean* by "shape?" To approach the question, then, one must put it differently, namely, in what terms

should one speak about the shape of a conic section. To this, Book VI provides two different answers, one in terms of coincidence, both of whole conic sections and parts (or segments) of conic sections, and another in terms of lengths and widths, again both in whole conic sections and parts; these are, of course, equality and similarity, respectively. And it is perhaps unnecessary to say that this is an essentially geometric investigation carried out in a completely geometric manner. In asking about shape, one can direct the question towards *two* conic sections and ask whether they have the *same* shape, or towards an *individual* conic section. It is not always easy to separate these. C. M. Taisbak has suggested the former is characteristic of the *Elements* while the latter, of the *Data.* Having considered whether *Data* 41 and *Elem.*VI.6 are what Knorr calls "parallel results," Taisbak writes, "The *Elements* compare triangles. The *Data* deals with individuals, and with the 'knowledge' we may have of such individual triangles within the language of givens. A superfluous discipline? Rather, a useful one in geometric analysis, where individual objects prevail."[77] Taken without the constructions that conclude the book, the greater part of Book VI, which speaks explicitly of two parabolas, hyperbolas, or ellipses, or of a parabola and a hyperbola, and so on, certainly appears to be, in this connection, more like the *Elements* than the *Data.* But the constructions redirect our attention again towards individual conic sections. And the prominent use of the word "given" in the constructions does call one's attention to the *Data* and its theoretical inquiry,[78] without forcing one to conclude that Book VI is, in fact, a book about δεδομένα rather than κωνικαὶ τομαί.

The abilty to call a conic section "given" is, in a sense, concomitant with the ability to *separate* mentally the conic section from the cone. And we suggest here that it is in Book VI only that there is an attempt to do this. If this is correct, then it gives further evidence for what we have argued elsewhere, namely, that the συμπτώματα do not *define* conic sections, or, at least, Apollonius does not assume that they do. Surely, the συμπτώματα play a part in Book VI, but it is only when Apollonius shows that one can *begin* with a conic

[77] Taisbak, "Elements of Euclid's *Data*," p. 142.

[78] Taisbak has said of the *Data*, "I take it to be a serious, if unsuccessful, attempt by some philosophical mathematician(perhaps the young Euclid) to penetrate into the foundations of geometry" ("Euclid's Data," p. 13).

and *end* with a cone that he is willing to see them as if separated from the cone and call them "given." Whether Apollonius was completely satisfied with his efforts, we will probably never know for sure. The very hesitancy with which he uses the word "given" in connection with conic section points, perhaps, to his *not* being completely satisfied.[79] But, be that as it may, the effort itself shows he thought this issue of whether, or how, a conic section could be "given" was worth addressing. Now, it is important to realize that this kind of issue, as well as the general issue of shape that brought us to individual sections in the first place, is not one that necessarily provides a context for impressive techniques and procedures such as one finds in Book V or Book III. In thinking about Book VI, one should perhaps think of the Apollonius who writes books such as the, so called, *General Treatise* (καθόλου πραγματείᾳ), about which Heath says "It would appear that this work must have dealt with the fundamental principles of mathematics, definitions, axioms, &c., and that to it must be referred the various remarks on such subjects attributed to Apollonius by Proclus, the elucidation of the notion of a line, the definition of a plane and solid angles, and his attempts to prove the axioms . . ."[80]

Quoting Heath above brings us back to the opening remarks of this chapter where we described the deprecatory tone with which Zeuthen and Heath refer to Book VI. It is telling that Heath's remarks about Book VI and those about the *General Treatise* are in the same chapter of the same volume. Despite his awareness of theoretical works such as the *General Treatise*, Heath does not seem to place too much weight on them, as if they were only of secondary importance to Apollonius. By contrast, he is full of praise and awe of Book V of the *Conica*, about which he writes, "Included in it are a series of propositions which, though worked out by the purest geometrical methods, *actually* [emphasis added] lead immediately to the determination to the evolute of each of the three conics; that is to say, the Cartesian equations of the evolutes can be easily deduced from the results obtained by Apollonius. There can be no doubt that

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[79] We point out that, while one finds a circle or arc of a circle "given" in Greek analyses, one finds few instances in which a *conic section* is said to be "given." This may mean that the "givenness" of conic sections, and other higher curves, remained always a thorny issue.

[80] Sir Thomas Heath, *A History of Greek Mathematics*, II, 192–3.

the Book is almost wholly original, and it is a veritable geometric *tour de force*."[81] It seems that what impresses Heath is a kind of technical brilliance, especially when it can be connected (by him . . .) to what would be considered an advanced topic in a modern classroom, such as the evolute. This tendency goes hand in hand with the conviction on the part of Heath and, of course, Zeuthen, whose thought was clearly a tremendous influence on Heath,[82] that an algebraic setting is an appropriate one for speaking about the *Conica.*[83] Indeed, from an algebraic standpoint the accomplishments of Book VI are not very striking. For example, if a curve is given by the coordinates (x,y) of the points in the plane through which the curve passes, and another curve (in a similar position with respect to the coordinate system) by the coordinates (x',y'), then the two curves will be similar if there is a single number k such that x = kx' and y = ky'. Let there be two hyperbolas or ellipses whose equations are (1) $x^2 = py\left(1 \pm \frac{y}{d}\right)$ and (2) $x'^2 = p'y'\left(1 \pm \frac{y'}{d'}\right)$ (in the sign "±", "+" corresponds to the hyperbola and "–" to the ellipse), where p and p', d and d' are the respective *latera recta* and diameters. Now, suppose (3) $\frac{p}{p'} = \frac{d}{d'} = k$, so that the figures of the hyperbolas or ellipses are similar. We will show that the hyperbolas or ellipses are similar. In

[81] *Ibid.*, II,159. See also Heath, *Apollonius of Perga: Treatise on Conic Sections*, p. vii.

[82] Heath does not deny this. He writes, "For the purposes of the earlier history of conics, and the chapters on the mathematical significance of certain portions of the *Conics* and of the other smaller treatises of Apollonius, I have been constantly indebted to an admirable work by H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum* (German edition, Copenhagen, 1886), which to a large extent covers the same ground, though a great portion of his work, consisting of a mathematical analysis rather than a reproduction of Apollonius, is of course replaced by the re-edited treatise itself" (*Apollonius of Perga: Treatise on Conic Sections*, p. xi).

[83] The relationship between an understanding of mathematics as algebraic and a view of mathematics as a technical subject has a basis deep in the history of algebra. J. Winfree Smith has written in his "Preface" (unpublished) to *Viète's Introduction to the Analytical Art* that ". . . Viète identified mathematical science with mathematical art. Plato had distinguished the science of arithmetic, i.e., the knowledge of numbers and of the several classes into which they fall, from the arithmetical art which he called "logistic (λογιστική τέχνη)." The latter has to do with calculating, with the addition, subtraction, multiplication, or division of numbers and with computing by means of ratios (λόγοι) of numbers. Logistic itself may be either theoretical when the calculations have to do with numbers of pure units or practical when they have to do with numbers of sheep or acres or the like. Now Viète recognizes no distinction between, on the one hand, mathematical science in the sense of knowledge of mathematical objects and on the other, mathematical art" (p. 8). See also Klein, *Greek Mathematical Thought and the Origin of Algebra*, pp. 121–122.

particular, we will show that the equations (1) and (2) are related by the equations x = kx' and y = ky'. Subsituting kx' for x and ky' for y in (1), we have $k^2x'^2 = pky'\left(1 \pm \frac{ky'}{d}\right)$. Divide both sides by k^2 to obtain, $x'^2 = \frac{p}{k}y'\left(1 \pm \frac{ky'}{d}\right)$. By (3), however, $\frac{p}{k} = p'$ and $\frac{k}{d} = \frac{1}{d'}$, so the last equation becomes equation (2). So much for VI.12 (as well as VI.2 where k = 1). In the third section above ("Equality and Similarity"), we attacked this approach on the grounds that it could not explain Apollonius's so clearly distinguishing equality and similarity. But if one *were* to accept this algebraic interpretation, the view of Book VI as a mere expansion of ideas implied in Book I for the "less sure investigator"[84] would be almost unavoidable; certainly similarity would present no special conceptual difficulties, and whether a conic section is "given," and, therefore, separable in some way from the cone would be a non-issue since, in the algebraic view, right from Book I a conic section is given by its equation. Yet, one such as Heath could not have missed the fact that in the letter prefacing Book I, which is also the general introduction to the *Conica*, Apollonius *himself* gives no less emphasis to Book VI than, say, to Book V. This is not meant to belittle the achievements in Book V (Heaven forbid!) but only to stress that for Apollonius,at least, Book VI has a no less justified place than any other among the books of the *Conica*. The sort of theoretical issues which we have seen above and which secure its place in the *Conica*, are manifest only when one is willing to remove all modern, and especially algebraic, preconceptions from one's reading of Book VI.

[84] See note 3 above.

Figures

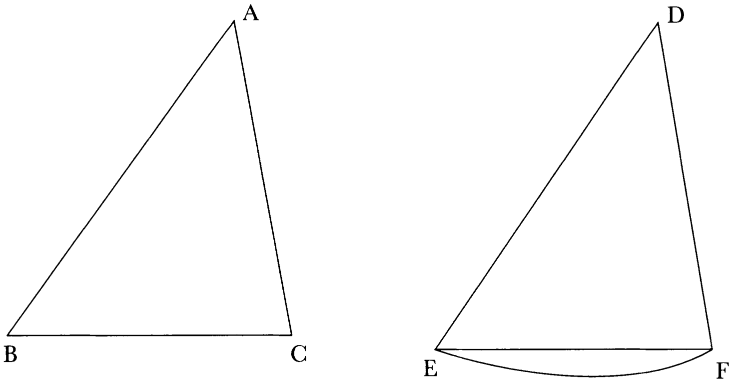
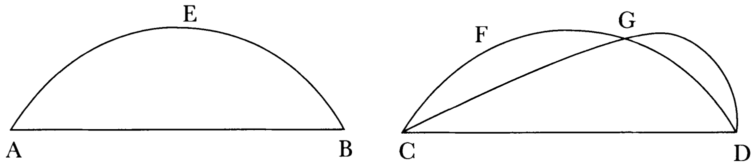


Fig. 1



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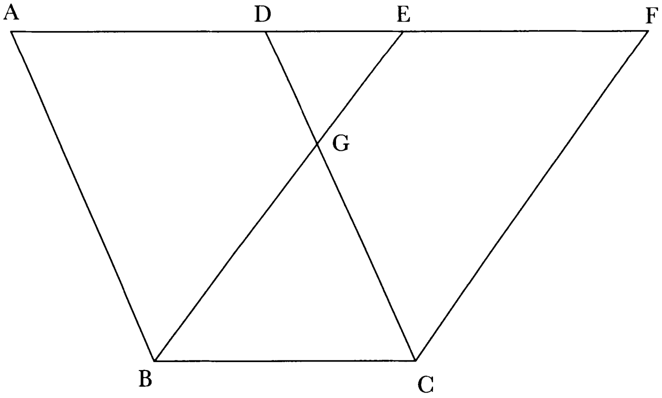
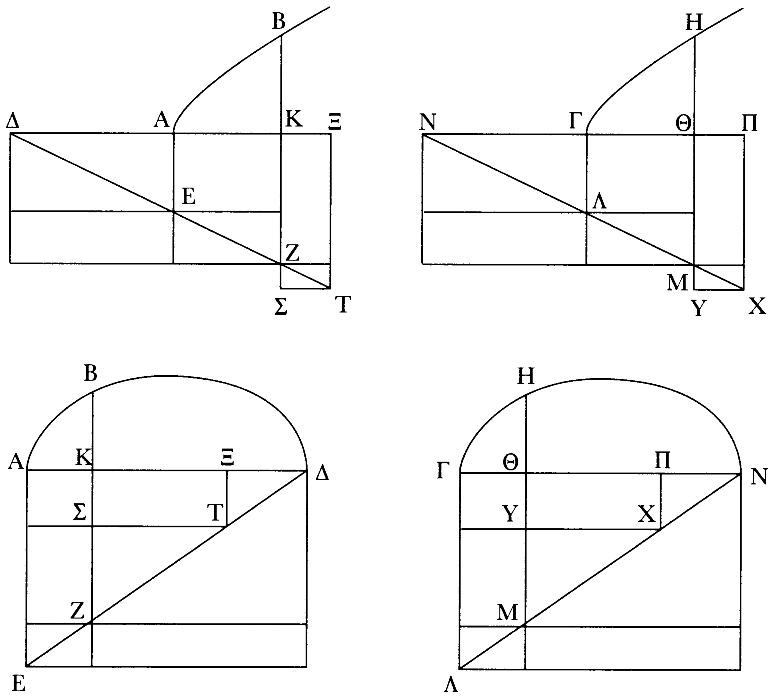


Fig. 3



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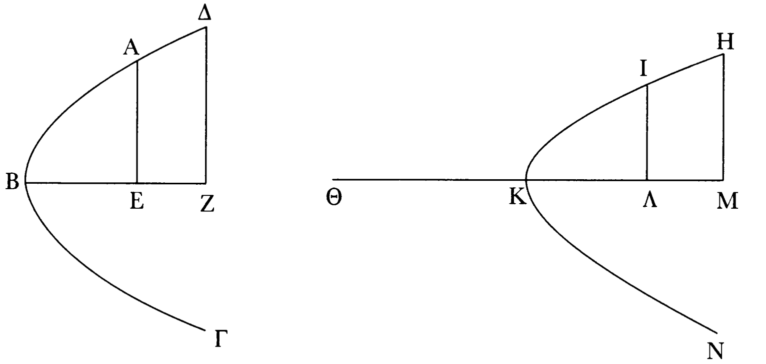

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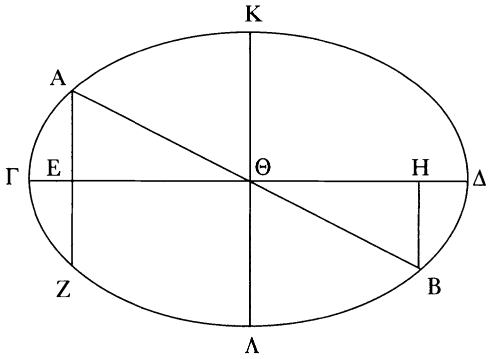

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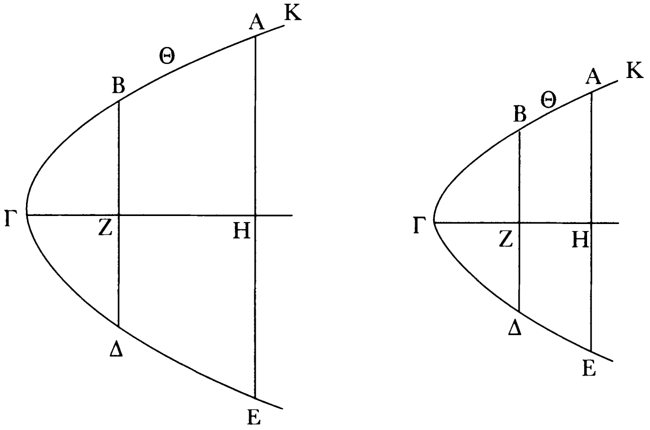

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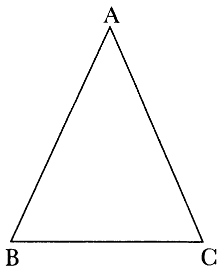

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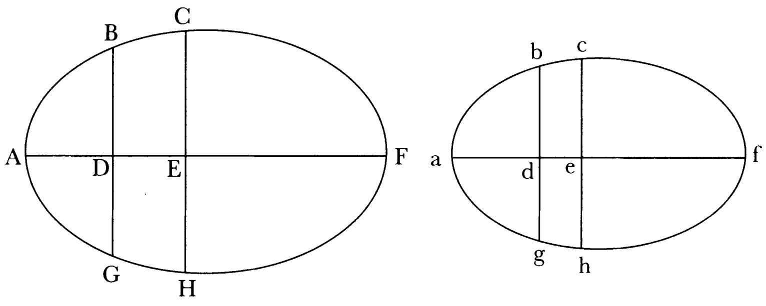


Fig. 9

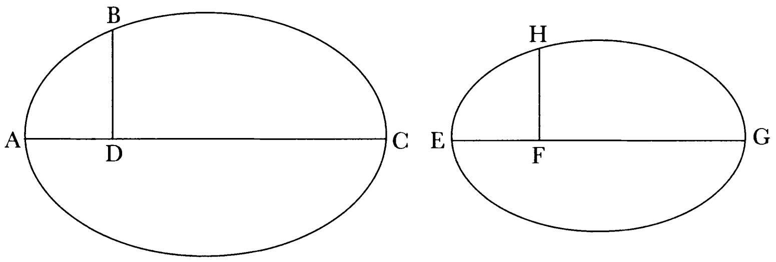
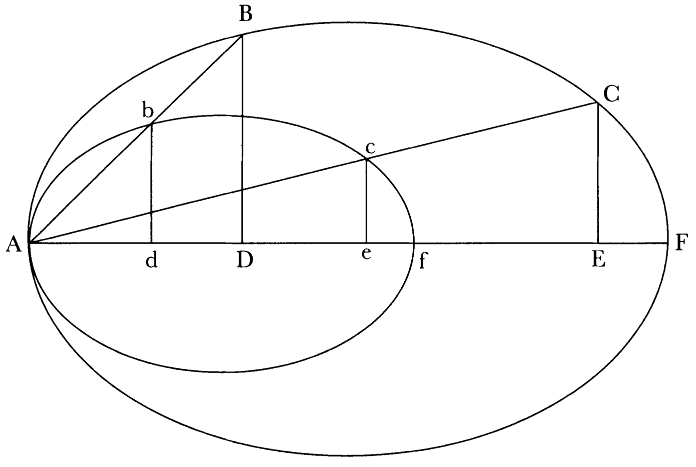


Fig. 10



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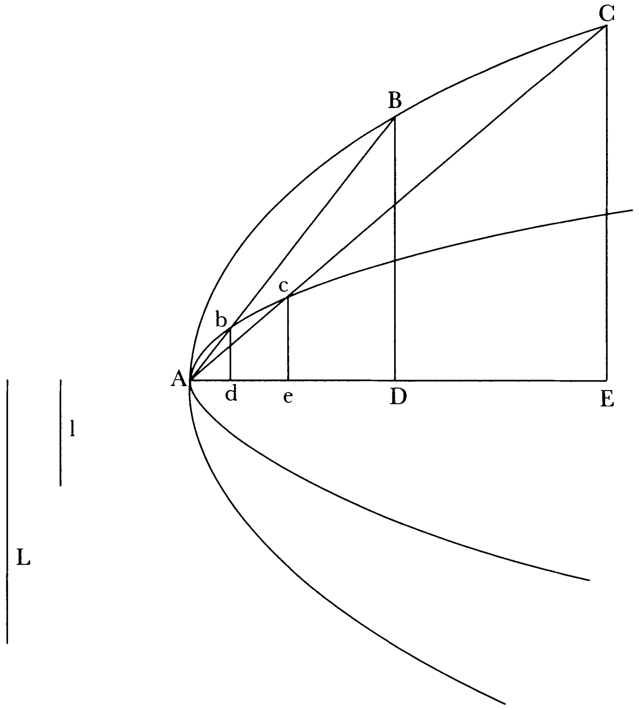
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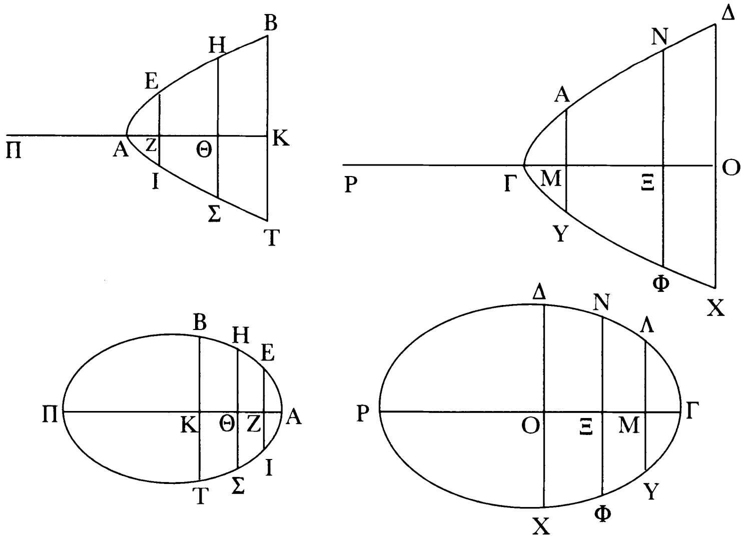
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Fig. 13

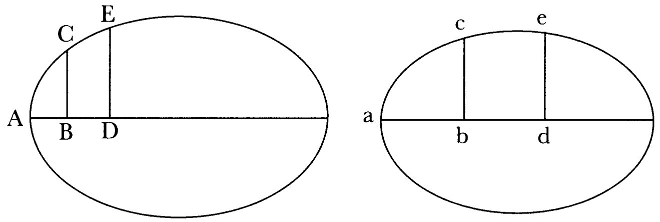

Fig. 14

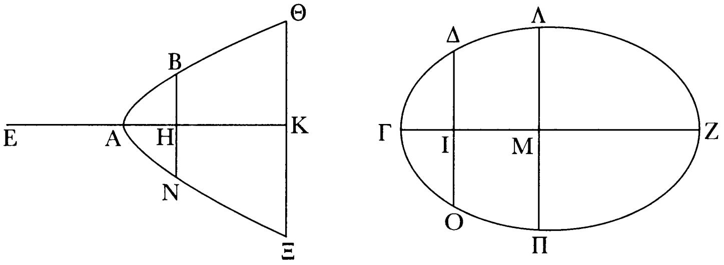

Fig. 15

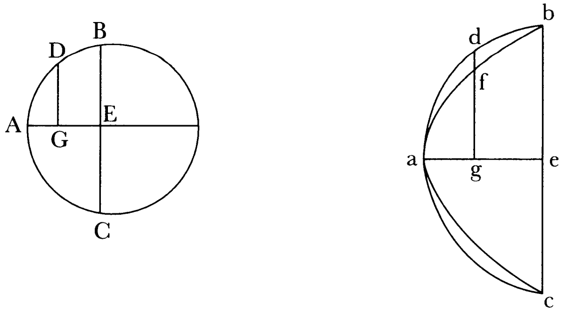

Fig. 16

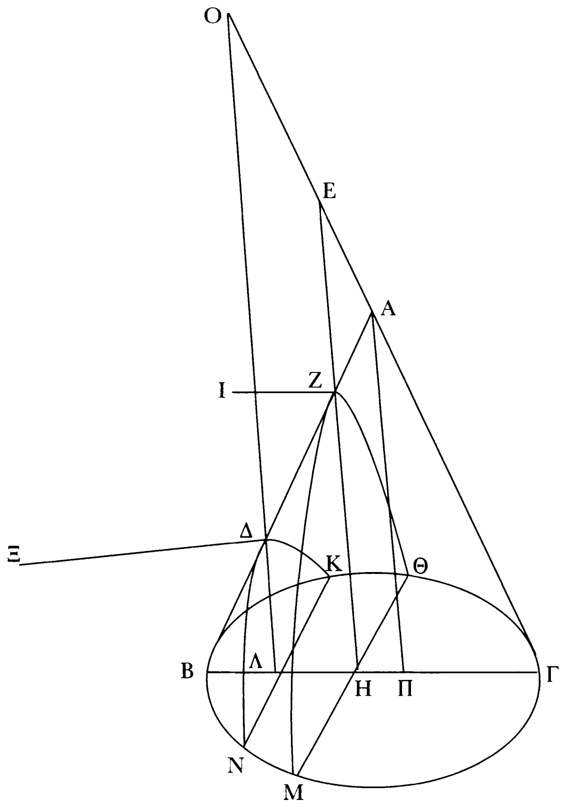

Fig. 17

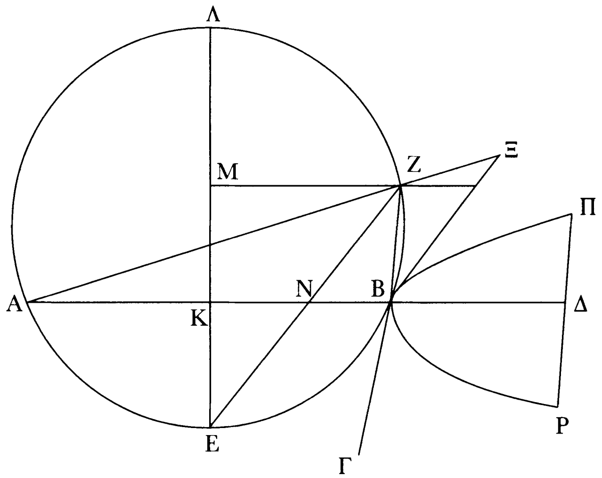

Fig. 18

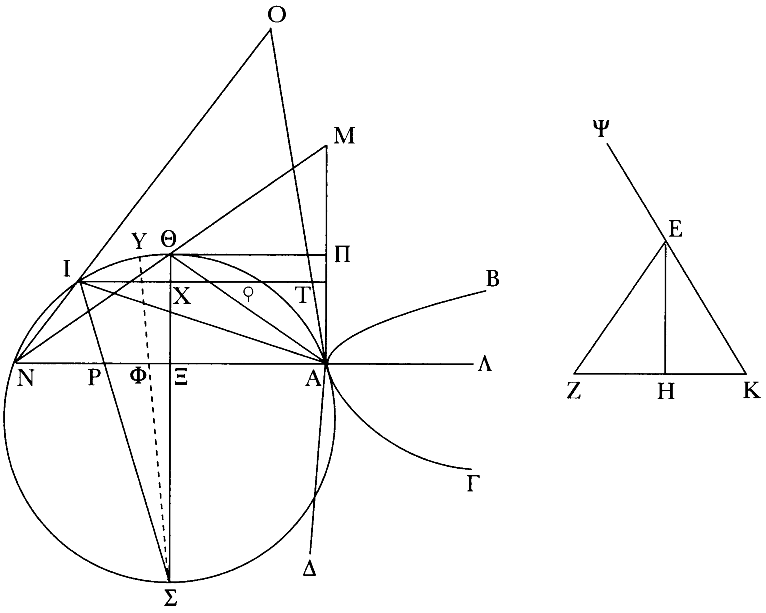

Fig. 19

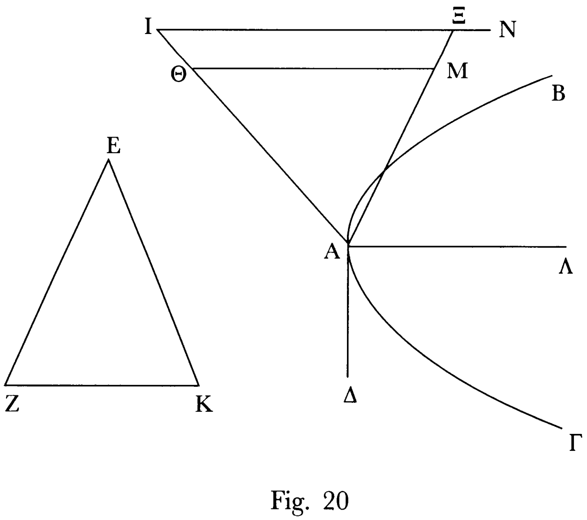

Fig. 20

CHAPTER SIX

DIORISMIC THEOREMS AND THE USE OF ANALOGY: BOOK VII OF THE *CONICA*

Introduction

Apollonius describes the contents of Book VII, the last of the extant books of the *Conica*, in two ways. In the general introduction opening Book I, he says regarding Book VII only that it concerns *diorismic theorems* (περὶ διοριστικῶν θεωρημάτων), or, as the phrase is often translated, "limiting theorems."[1] In the prefatory letter to Book VII itself, however, Apollonius tells us that the book contains ". . . many wonderful and beautiful things on the topic of diameters and the figures constructed on them." Apollonius adds in the latter that such things are of great use in problems involving conic sections, including those of the kind to be discussed in Book VIII. One cannot say, therefore, that the two descriptions are inconsistent. Nevertheless, one does wonder why Apollonius's description of Book VII in the general introduction fails to mention diameters and the figures constructed on them. Conversely, one wonders why Book VII is *singled out* as a book concerning *diorismic theorems*. Indeed, Apollonius points out that propositions in Books II, III, IV, and V are also useful in the construction of *diorismoi*, so why should Book VII acquire a *special* status in this regard? To begin to answer these questions, we shall have to understand better what is meant by a *diorismic theorem* and the way propositions in Book VII are theorems of that kind. This, then, is the purpose of the first part of this chapter.

In the course of the discussion, we shall have to consider the relationship between *diorismic theorems* and *diorismoi*; in particular, we shall ask whether *diorismic theorems* must be tied to particular problems, or whether they are, rather, the kind of theorems only useful for the construction of *diorismoi* of the solutions of problems. To this end,

[1] Thus, for example, Talieferro translates the phrase. I. Thomas writes "theorems about the determination of limits," which, as we shall explain later, we take to be a less felicitous translation of "περὶ διοριστικῶν θεωρημάτων."

we shall look at some propositions in Euclid's *Elements*, as well as some other propositions in the *Conica*; we shall also, in this context, look briefly at Halley's reconstruction of Book VIII. Following that, we shall look at the *diorismic theorems* as independent theorems and consider what they add to our actual knowledge of conic sections.

While the first part of this chapter, accordingly, concerns the *subject* of Book VII, the second part concerns its dominant *procedure*. The most salient aspect of that is the use in Book VII of what Apollonius calls the *homologue*. The striking presence of the *homologue* has been observed, for example, by Heath. However, in their desire to find "... the geometrical substitute for algebraical operations,"[2] we believe that Heath and others like him miss the true significance of the *homologue*. For its use, as we shall show, is to be found not only in the setting out of a proportion, an *analogia*, which lends itself, more or less readily, to algebraic formulations, but also in the making of *likenesses*, and this involves a kind of thinking quite different from that guided by algebra. Indeed, Book VII is replete with likeness-making. And such likeness-making, starting with Apollonius's use of the *homologue*, reflects, in a particularly clear way, the presence of *analogical* thinking in the *Conica*. Thus, whereas in previous chapters we have often pointed to *particular* analogies occurring in Apollonius's work, in the present chapter we shall begin a discussion of analogy *as a genuine mode of thinking* about conic sections (the full development and completion of this discussion we shall leave to our chapter on "Conics and Circles"). For this reason, we ought to remark, if it is not already clear, that although it has been convenient to consider the subject of Book VII in one part and its procedure in another, the separation of subject and procedure is, of course, somewhat artificial; the analogical thinking represented in Book VII is yet another expression of Apollonius's geometrical way of conceiving the object of his work.

Diorismoi and Diorismic Theorems

As we mentioned above, Apollonius on several occasions mentions the relevance of propositions in the *Conica* to *diorismoi*, or "limits of

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[2] Heath, *Apollonius of Perga*, p. cxiii. The quotation refers, specifically, to an important proportion proven in VII.6,7 in which the ratio of the squares on conjugate

possibility." In the prefatory letter to Book I, the general introduction to the whole work, he says of Book II that "it contains the properties having to do with the diameters and axes and also the asymptotes, and other things *of a general and necessary use for limits of possibility* (γενικὴν καὶ ἀναγκαίαν χρείαν παρεχόμενα πρὸς τοὺς διορισμοὺς) [emphasis added]."[3] Immediately following this, he says about Book III that it ". . . contains many incredible theorems of use for the construction of solid loci and for limits of possibility . . ." Similarly, in the introduction to Book V, Apollonius says regarding the propositions about maximum and minimum lines that ". . . students of this science need them for the knowledge of analysis and determination of problems [by which is meant *diorismoi*, as Toomer rightly points out][4] and their synthesis . . ." In the prefatory letter to Book IV Apollonius refers, at some length, to the relevance of the propositions in that book to *diorismoi*: "The investigation [of these theorems] is also of considerable use in the synthesis of problems and limits of possibility. So, Nicoteles was not speaking truly when, for the sake of his argument with Conon, he said that none of the things discovered by Conon were of any use for limits of possibility; but even if the limits of possibility are able to be obtained completely without these things, yet, surely, some matters are more readily perceived by means of them, for example, whether [a problem] might be done in many ways, and in how many ways, or again, whether it might not be done at all."

But what is a *diorismos*? Let us review the notion. There are, as Eutocius tells us is "clear to everyone everywhere,"[5] *two* mathematical contexts in which the term *diorismos* is used. The first is in the classification of the parts of a proposition. Proclus tells us that whereas ". . . the enunciation states what is given and what is being sought from it . . .," the *ekthesis* or "exposition takes separately what is given and prepares it in advance for use in the investigation and then the *diorismos* or "specification takes separately the thing that is sought

diameters is shown to be the same as a ratio of lines connected with the *homologues.* We shall discuss VII.6,7 in some detail later.

[3] As always, we are using Taliaferro's translation for quotations from Books I–III, Toomer's translation for quotations from Books V–VII, and our own for quotations from Book IV.

[4] Toomer, *Apollonius* Conics *Books V to VII,* p. 551.

[5] Eutocius, *In Conica,* p. 178.

and makes clear precisely what it is."[6] What "preparation" the *ekthesis* carries out is to set out objects given in the enunciation in *particular* terms, that is, where the enunciation will speak of "a point," "a line," "a triangle," the *ekthesis* will speak of "point A," "line AB," "triangle ABC"; the *diorismos* then, *in these terms*, states what is to be proved. The word *diorismos*, of course, is derived from the verb διορίζω, "to draw a boundary through," "to define," "to distinguish," and this, in turn, is derived from ὅρος, "a boundary" or "definition." So, once the *ekthesis* has set out the particular object to which the proof will refer, the *diorismos defines* what is to be proved for the particular object; it says the proposition will be proved *within the bounds* of this particular object or these particular givens. In effect, the *diorismos restricts* our attention to what occurs *within the bounds* of the diagram. Naturally, it does not restrict the givens themselves, that is, the particular objects do not transform a general proposition into a mere special case. Thus, Proclus writes only that "In a sense the purpose of the specification is to fix our attention; it makes us more attentive to the proof by announcing what is to be proved, just as the exposition puts us in a better position for learning by producing the given element before our eyes."[7]

Understanding a *diorismos* as something that restricts or bounds in some way leads us to the second context in which the notion of *diorismos* is used, namely, that of problems, where it is thought of as a "limit of possibility." What *diorismos* means in this latter context is stated by Eutocius who tells us that, "when a proposition (πρότασις) cannot in general be granted (συγχωρῶν)," the *diorsmos* determines exactly "when and how and in how many ways what is put forth can be done (συστῆναι)."[8] This echoes Apollonius's own statement at the end of the quotation cited above from Book IV, namely, that

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[6] *In Eucl.*, p. 203 (Morrow's translation, p. 159).

[7] *Ibid.*, p. 208 (Morrow, p. 163). The *ekthesis* and *diorismos* are emblematic of the Greek tendency to work with the concrete and the visible, as we discussed earlier and as has been discussed elsewhere in the literature (e.g. in Netz, *The Shaping of Deduction, passim*). In this sense, then, Mueller's saying that the repetition of the fact to be proved in the *ekthesis-diorismos* (and again in the conclusion of the proof, in the *sumperasma*) "would seem to be connected with the *difficulty* [emphasis added] of grasping the idea of generalization" (*Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, p. 13) seems to us misleading.

[8] *In Conica*, p. 178. See also Pappus, *Collection*, VII, p. 636: "Diorismos is the preliminary distinction (προδιαστολή) of when, how, and in how many ways the problem is possible."

diorismoi determine "... whether [a problem] might be done in many ways, and in how many ways, or again, whether it might not be done at all." While these descriptions seem to allow for the *diorismos* to be *any* kind of condition for the solvability of a problem, the *diorismos* usually refers to numbers, continuous magnitudes, or ratios insofar as they can be ordered in some way, that is, so that one can speak about a number, figure, or ratio being greater or less, greatest or least;[9] thus, in general, the *diorismos makes a restriction* or *gives bounds* as to how much, how small, how many.[10]

As an example of a proposition containing a *diorismos*, and it is a standard example, Eutocius recalls *Elem.*I.22 where the *problem* is to construct a triangle from three straight lines equal to three given straight lines, and the *diorismos* is that it be necessary, then, that two of the straight lines taken together in any way be greater than the remaining one.[11] An example from the *Conica* of a *diorismos* in this standard form is in VI.32, in which Apollonius says: "We want to show how to construct a right cone, similar to a given right cone,

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[9] About *diorismoi*, Toomer says that "Since, generally speaking, the only solutions to problems recognized in ancient mathematics were what we should term as those embodied in real, positive numbers, it frequently happened that a problem could be solved only if certain boundary conditions were satisfied, or that the number of solutions would vary with the conditions. The διορισμός consisted in the determination of those boundary conditions and of the number of solutions" (*Apollonius* Conics *Books V to VII*, pp. lxxxiv–lxxxv). This gives a single characterization for a wide variety of *diorismoi*. However, it is also an anachronistic one. For it relies on the homogenizing potency of modern real numbers; it effaces the important distinctions in Greek mathematics between continuous magnitudes, numbers, and ratios.

[10] Possible exceptions are conditions of the kind found in *Elem.*IX.18, where it is stated when two given numbers can have a third proportional (namely, that they not be prime to one another and that one measures the square on the other) and in *Elem.*IX.19, where, similarly, it is stated under what conditions three given numbers can have a fourth proportional (see Vitrac, *Euclide*, II,440–443).

[11] An analogous condition given in the *Elements* is XI.22 which states: "If there be three plane angles of which two, taken together in any manner, are greater than the remaining one, and they are contained by equal straight lines, it is possible to construct a triangle out of the straight lines joining the extremities of the equal straight lines." Knorr counts this as "A Euclidean example" of a *diorismos* (*Ancient Tradition of Geometric Problems*, p. 336), but that must be claimed with some caution: the *diorismos* usually follows the statement of the problem, not as a given for the problem but as a genuine condition for its solvability. Such, for example, is Proposition XI.23, which Knorr also mentions: "To construct a solid angle out of three plane angles two of which, taken together in any manner, are greater than the remaining one [*the problem*]: thus, the three angles must be less than four right angles [*the diorismos*]. Proposition XI.22, however, may be termed a *diorismic theorem* (διοριστικόν θεώρημα), for it states a fact potentially useful in framing *diorismoi*; this distinction between *diorismos* and *diorismic theorem* we shall discuss below.

containing a given hyperbola [*the problem*]*. [For this problem to be soluble] it is necessary that the ratio of the square on the axis of that cone to the square on the radius of its base be not greater than the ratio of the transverse diameter of the figure constructed on the axis of the section [the given hyperbola] to its *latus rectum* [*the diorismos*]*."[12] The *diorismos*, in this case, is proven explicitly in the course of the proof of the proposition as a whole: with ΒΑΓ as the given hyperbola (see fig. 1), having transverse diameter ΑΝ and upright side ΑΔ, and ΕΖΚ the given cone, having ΕΗ as its axis and ΖΗ the radius of its base, Apollonius says that the problem will not be solvable whenever sq.ΕΗ:sq.ΖΗ *exceeds* ΝΑ:ΑΔ. In the initial construction, Apollonius had set up a circle ΝΘΑ in the plane perpendicular to that containing the given section and such that the segment ΝΘΑ contained an angle equal to ΖΕΨ, the angle complementary to that at the vertex of the given cone. Accordingly, any right cone such as that having its vertex at point Ι and its base the circle whose diameter is ΑΟ would be similar to the given cone. So suppose sq.ΕΗ:sq.ΖΗ > ΝΑ:ΑΔ and that the given hyperbola were contained by a cone such as that just described. "Then we will prove by a method like the preceding that ΣΡ:ΡΙ = ΝΑ:ΑΔ [where ΣΞΘ bisects arc ΝΘΑ and is perpendicular to ΝΑ]*.

> But ΝΑ:ΑΔ < sq.ΕΗ:sq.ΖΗ, which we proved [earlier in the proof]* to be equal to ΣΞ:ΞΘ.
> Therefore, ΣΡ:ΡΙ < ΣΞ:ΞΘ: that is absurd.
> So no cone [of this kind] similar to cone ΖΕΚ will contain section ΒΑΓ.

This example thus shows to what extent a *diorismos* is an organic part of the problem; indeed, in VI.32, the *diorismos* is as much a conclusion within the demonstration of the problem as it is a condition for it. Nevertheless, it *is* still a condition because it is directed towards the *givens* of the problem.[13] In this connection, one ought to observe that within VI.32 there are other conclusions as well which one can take to be related to *diorismoi*, that is, conclusions which determine ". . . whether [a problem] might be done in many ways, and in how many ways, or again, whether it might not be done at

[12] As usual the quotations from Books V to VII are in Toomer's translation *Apollonius* Conics *Books V to VII*, and, as before, bracketed phrases and words in which the brackets are followed by an asterisk are our addition.
[13] See Knorr, *The Ancient Tradition of Geometric Problems*, p. 73.

all." For Apollonius shows that there is *one solution* to the problem *and no more* whenever NA:AΔ is the same as sq.EH:sq.ZH, and there are *two solutions* to the problem *and no more* whenever NA:AΔ is greater than sq.EH:sq.ZH.[14]

So much for the *diorismoi* in problems, but what about *diorismic theorems*? What are *they*? After all, in the general introduction to the *Conica*, it is *diorismic theorems* that Book VII is said to contain, not *diorismoi*. To begin to answer this, let us consider a pair of propositions from Euclid's *Elements*, *Elem.*VI.27,28, which are usually associated with the subject of *diorismoi*.[15]

*Elem.*VI.28 is enunciated in the standard form, namely, the problem whose solution is to be constructed followed by its *diorismos*: "To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammic figure similar to a given one [*the problem*]: thus the given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect [*the diorismos*]."[16] In the course of the demonstration, it becomes clear that the *diorismos* not only limits the solution but that it also plays an important part in guiding the construction of the solution. Here is a partial paraphrase of Euclid's solution: Let AB be the given straight line (see fig. 2), C the given figure to which the applied parallelogram must equal, and D the parallelogram to which the defect must be similar. Let AB be bisected at E and let parallelogram EBFG be described on EB similar and similarly situated to D, and let parallelogram AG be completed. Clearly, if C is equal to AG, then the problem is solved. "But, if not", Euclid continues," let HE be greater than C.

[14] It should be noted that Apollonius does not indicate *these diorismoi* in the enunciation, but only that which determines whether or not the problem is solvable. This has to do not with Apollonius's understanding of *diorismoi*, but with his ends in Book VI, which, as we argued in the last chapter, are directed towards the problem of how conic sections *themselves* can be *givens*. Thus, in the constructions at the end of Book VI, Apollonius's primary goal is to show that *a* conic section can be provided equal or similar to a given one; it is less important at this juncture to know in how many different ways this can be done.

[15] In light of our chapter on geometrical algebra, we see no need here to consider the algebraic reading of these propositions where they are interpreted as concerning the quadratic equation $Mx - Nx^2 = L$, and, in particular, where the *diorismos* is interpreted as the algebraic condition $M^2 > 4NL$. A detailed and specific analysis of the inherent inadequacies of the algebraic interpretation of VI.27,28 appears in Unguru and Rowe, "Does the Quadratic Equation have Greek Roots?"

[16] As always, unless otherwise noted, quotations from the *Elements* are in Heath's translation.

Now HE is equal to GB; therefore GB is also greater than C.

Let KLMN be constructed at once equal to the excess by which GB is greater than C and similar and similarly situated to D.[17]

But D is similar to GB; therefore KM is also similar to GB.[18]

Let, then, KL correspond to GE, and LM to GF.

Now, since GB is equal to C, KM, therefore GB is greater than KM; therefore also GE is greater than KL, and GF than LM.

Let GO be made equal to KL, and GP equal to LM; and let the parallelogram OGPQ be completed; therefore it is equal and similar to KM.

Therefore GQ is also similar to GB; therefore GQ is about the same diameter with GB.[19]

Let GQB be their diameter, and let the figure be described.

Then, since BG is equal to C, KM, and in them GQ is equal to KM, therefore the remainder, the gnomon UWV, is equal to the remainder C.

And from here the proof follows lines familiar to anyone acquainted with Book II of the *Elements*: since parallelogram PR is equal to parallelogram OS, parallelogram PB is equal to parallelogram OB; parallelogram OB is equal to parallelogram TE, so that also parallelogram PB is equal to parallelogram TE; therefore, parallelogram TS is equal to the whole gnomon UWV, which is equal to C. Thus, the problem has been solved since TS is a parallelogram equal to C which has been applied to AB and deficient by a parallelogram QB which is similar to D.

Clearly, this solution could not have been effected had the *diorismos* not required the figure C to be less than or equal to parallelogrammic figure GB, for otherwise one obviously could not consider the *excess* of GB over C, which, of course, is the key to the solution. But does the *diorismos* in *Elem.*VI.28 truly limit the *solution* or only the *particular procedure* provided in the demonstration? In other words, if one starts out to solve the problem of applying a parallelogram to a given line equal to a given figure and falling short by a figure similar to a given parallelogram by bisecting the given line and

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[17] By *Elem.*VI.25, which shows how to construct a rectilinear figure similar to a given figure and equal to another given figure.

[18] By *Elem.*VI.21, which proves that two figures similar to the same figure are similar to one another.

[19] By *Elem.*VI.26, which proves that "If from a parallelogram there be taken away a parallelogram similar and similarly situated to the whole and having a common angle with it, it is about the same diameter with the whole."

describing a parallelogram on the half similar to the given parallelogram, then one will arrive at a solution only if the given figure is less than or equal to that parallelogram described on the half. But might it not possible that with some *other* procedure one could relax this constraint? The answer to this is given by *Elem.*VI.27, which states that "Of all the parallelograms applied to the same straight line and deficient by parallelogrammic figures similar and similarly situated to that described on the half of the straight line, that parallelogram is greatest which is applied to the half of the straight line and is similar to the defect." This shows that in fact the constraint given in the *diorismos* to *Elem.*VI.28 cannot be relaxed.[20] But more than that, *Elem.*VI.27 also sets the pattern for the actual solution of the problem in *Elem.*VI.28. Consider the demonstration for *Elem.*VI.27:

> Let AB be a straight line and let it be bisected at C (see fig. 3); let there be applied to the straight line AB the parallelogram AD deficient by the parallelogrammic figure DB described on the half of AB, that is, CB;
>
> I say that, of all the parallelograms applied to AB and deficient by parallelogrammic figures similar and similarly situated to DB, AD is greatest.
>
> For let there be applied to the straight line AB the parallelogram AF deficient by the parallelogrammic figure FB similar and similarly situated to DB; I say that AD is greater than AF.
>
> For, since the parallelogram DB is similar to the parallelogram FB, they are about the same diameter [by *Elem.*VI.26].
>
> Let their diameter DB be drawn, and let the figure be described.
>
> Then, since CF is equal to FE, and FB is common, therefore the whole CH is equal to the whole KE.
>
> But CH is equal to CG, since AC is also equal to CB.
>
> Therefore GC is also equal to EK.
>
> Let CF be added to each; therefore the whole AF is equal to the gnomon LMN; so that the parallelogram DB, that is, AD, is greater than the parallelogram AF.[21]

This very close relationship between *Elem.*VI.27 and *Elem.*VI.28 leads one either to think of *Elem.*VI.27 as a lemma for *Elem.*VI.28 or

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[20] This is obvious by a simple analysis: assuming the problem has been solved and QB is the defect similar to D, then TS is equal to C. But by *Elem.*VI.27, TS cannot exceed AG, therefore, C cannot exceed AG.

[21] Euclid's proof takes into account only the case where the defect FB is *less* than the similar parallelogram on the half of AB, DB. A slight variation of the proof, however, easily disposes of the case where FB is greater than DB (see Heath, *Euclid*, II, 260).

*Elem.*VI.28 as an application of *Elem.*VI.27. Given the consistency of its theme with the rest of Book VI and its own intrinsic interest,[22] it is unlikely that *Elem.*VI.27 was meant by Euclid to be a mere lemma for *Elem.*VI.28. It is more reasonable to view *Elem.*VI.28 as a problem applying the ideas from the previous propositions—not only from *Elem.*VI.27, but also from *Elem.*VI.21,25, and 26. Yet, though more reasonable, this possibility too is not altogether satisfying. For while *Elem.*VI.21,25, and 26 are crucial in justifying the steps of the proof of *Elem.*VI.28, *Elem.*VI.27 is the very basis for the *diorismos*, without which the enunciation of the problem itself would be incomplete. In this way, *Elem.*VI.27 appears to have a different status with regards to *Elem.*VI.28 from that of propositions 21, 25, and 26. On the one hand, *Elem.*VI.27 stands by itself as a theorem with its own interest, that is, it does not simply serve proposition *Elem.*VI.28, while, on the other hand, once *Elem.*VI.28 is stated, *Elem.*VI.27 immediately suggests what the *diorismos* must be, as if such a *diorismos* were anticipated by *Elem.*VI.27. The right characterization for *Elem.*VI.27, therefore, seems to be neither as a lemma nor as a completely independent theorem from which problems may be derived.

What *is* the right characterization is contained in the form of the proposition. *Elem.*VI.27 is a proposition that immediately presents the reader with a genuine limit of possibility: of all parallelograms satisfying some given property, one certain definable parallelogram is the greatest of all, that is, it is not possible for it to be exceeded. The proposition thus has the *form* of a restriction; it is ready to be used as *diorismos*. However, and this is the important point, *it is not itself a diorismos* for it is not part of a problem whose givens it restricts. This is why we say it only *anticipates* a *diorismos*, as it were. Such theorems that anticipate *diorismoi* by having the form of a restriction are, we believe, what Apollonius means by *diorismic theorems* (διοριστικὰ θεωρήματα). This distinction between *diorismoi* and *diorismic theorems* helps understand why Apollonius can refer to theorems being of use in the formation of *diorismoi* without telling us what the particular problems are to which the *diorismoi* belong. For, if we are

[22] For example, in the special case where the defect is a *square*, the proposition shows that of all rectangles of given perimeter (*viz.*, twice the line AB in VI.27), the square contains the greatest area.

right, he is speaking of theorems that are only of a certain kind rather than of theorems which, like lemmas, are set out only to have a specific place in a specific problem.[23]

Of course, this does not mean that it is fruitless to try and link propositions that Apollonius relates to *diorismoi* with specific problems, only that that link is not always essential for the interpretation of those propositions, just as modern mathematicians sometimes refer to theorems as the kind of results useful in physics, because of their form or because of their resemblance to other useful theorems, without necessarily having specific applications in mind. It would be worth our while, nevertheless, to take a brief look at an impressive attempt to make such a link between the theorems of Book VII and problems to which they might have applied. That attempt is Halley's reconstruction of Book VIII of the *Conica.*

A Glance at Halley's Book VIII

Halley's reconstruction of the lost Book VIII begins with the assumption that the contents of that book were closely related to material in Book VII. In his prefatory letter to Henry Aldrich,[24] Halley tells us that it was Aldrich who suggested that the last book of the *Conica* "could surely be restored, given the fact that in Pappus's *Mathematical Collection*, Pappus himself passed on lemmas for what was needed to be demonstrated in the seventh and the eighth book of the *Conica* at the same time; whereas for the other books, different [lemmas] are provided for the different [books]. Hence, to you [to Aldrich]

[23] Proclus tells us of a certain Leon who compiled a book of elements and who ". . . also discovered *diorismi*, whose purpose is to determine when a problem under investigation is capable of solution and when it is not" (*In Eucl.*, pp. 66–67, Morrow's translation, p. 55). We tend to think that here too what is probably meant are *diorismic theorems*; if the intention were *diorismoi proper*, then one would expect Proclus to say something to the effect that Leon demonstrated the solutions to problems *and their diorismoi*, for, as we put it earlier, the *diorimos* is an organic part of the problem. By *separating* the *diorismos* from the problem, one gives it autonomy—makes it a theorem in its own right—while maintaining a form that makes it recognizable as the kind of theorem that is immediately applicable in a problem, and this, as we have said, is precisely what we mean by a *diorismic theorem.*

[24] *Apollonii Pergaei Conicorum Libri Octo*, p. 137. Aldrich was dean of Christ Church, and it was he who initially suggested the project of producing a Latin translation of Apollonius's works to David Gregory and Halley.

we owe the discovery that the two books [Books VII and VIII] are conjoined; that the problems of the Eighth Book are given their determination through the *diorismic* theorems of the Seventh." That Pappus does not distinguish the lemmas to Book VII from those to Book VIII, hints clearly to the fact that Books VII and VIII were indeed closely connected. However, from Pappus's lemmas alone, this could not be said conclusively. Certainly, those lemmas, which are of a fairly general character, could hardly form the basis of a *reconstruction* of Book VIII; indeed, having Book VII before us, it is easy to see that had *that* book been lost it would be impossible to divine its contents solely by the lemmas in Pappus.[25] It is hard to see then how Aldrich's remark about Pappus could really give Halley hope of restoring the lost book.[26]

But whether or not the credit Halley gave to Aldrich for the suggestion that Book VIII could be restored on the basis of Book VII was ingenuous, Halley could indeed be confident in the project because of the introduction to Book VII, which he himself had translated. For there, as we mentioned at the beginning of this chapter, Apollonius says explicitly that "All of this [*i.e.*, what is demonstrated in the propositions of Book VII] is of great use in many types of problems, and there is much need for it in the kind of problems which occur in conic sections which we mentioned, among those which will be discussed and proven in the eighth book of this treatise (which is the last book in it)." Now, this does not say that Book VIII is completely based on or derived from Book VII, only that the propositions in Book VII are of use in the kind of problems presented and solved in Book VIII. Halley, nevertheless, carried out his reconstruction as if Book VIII contained applications of the propositions of Book VII exclusively.[27]

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[25] A similar comment is made by Hogendijk (*Ibn al-Haytham's* Completion of the Conics, pp. 43–44).

[26] Indeed, Hogendijk observes that only one of the lemmas in Pappus related to Book VIII can be connected with problems in Halley's reconstruction, and that in not a very direct way. From that, Hogendijk concludes, "So the lemmas of Pappus do not support Halley's suppositions about the probable contents of Book VIII" (*op. cit.*, p. 48). But it may also show that the supposition was indeed *Aldrich's*, and that Halley did not take it very seriously.

[27] Knorr seems to accept this when he writes that ". . . although Book VIII is lost, we learn that it consisted of problems applying the results from Book VII, and from this Halley was able to propose a *full* [emphasis added] reconstruction" (*Ancient Tradition of Geometric Problems*, pp. 313–314). We do not *necessarily deny* this, but we caution that Apollonius's remarks in the letter opening Book VII give no sure evi-

The propositions in Book VII concern the diameters, upright sides, and figures of conic sections, particularly those that have figures, namely, the ellipse and the hyperbola; it considers the squares on the diameters and upright sides, the sums, differences, and ratios of conjugate diameters and of the squares on them, the sums and differences of diameters and their upright sides and of the squares on them, and so on. Halley, accordingly, took these theorems and made them the basis of problems and their *diorismoi.*[28] Consider, for example, Halley's VIII.29:[29] "Given the sides of the figure of the axis of an hyperbola, it is proposed to find a diameter of the section, the square of which together with the square of its *latus rectum* makes a given sum." Here is Halley's analysis (see fig. 4):

> The same things being as described in the previous diagrams of the hyperbola [in our figure 4, AB is the hyperbola, AΓ its transverse axis, AΞ:ΞΓ::ΓN:NA:: transverse axis: *latus rectum* relative to the axis, ΓΛ is parallel to BK, and ΛM is perpendicular to AΓ], consider it done, and let BK be the diameter which we seek: therefore (by VII.19), the square of[30] AΓ, or the rectangle contained by NΓ and the sum of the axis and its *latus rectum*, to the rectangle contained by NΓ & MΞ, that is, the sum of the axis and its *latus rectum* to MΞ, is thus the proposed sum of the squares of BK and its *latus rectum* to the sum of squares of NM and MΞ.[31] But, applying this given sum of squares [*viz.* sq.BK +

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dence that Book VIII consisted of problems related *exclusively* to Book VII. We put off the discussion as to the general adequacy of Halley's reconstruction until our chapter treating the *Conica* as a whole. But for now, we shall only say that it is, at least, a reasonable and intelligent *partial* restoration, and the problems presented in Halley's work may, as Heath says, represent "the nearest conjecture as to their contents *which is possible in the present state of our knowledge* [emphasis added] (*Apollonius of Perga*, p. lxxxvi).

[28] For this reason it is not surprising that in his translation of the introduction to Book VII, Halley adds the words "especially in their determinations (διορισμοῖς)" after Apollonius writes "All this is of great use in many types of problems." Toomer rightly points out that this is done "without any textual justification" (*Apollonius* Conics *Books V to VII*, p. 595n1).

[29] *Conicorum Libri Octo*, pp. 164–165.

[30] Here and below, Halley uses ". . . quadratum ex . . .," which we are translating ". . . the square of . . ."

[31] Writing lr(AB) for the *latus rectum* relative to diameter AB, Halley's conclusion can be justified as follows:

1) NΓ:AN::AΓ:lr(AΓ), so, NΓ:(AN + NΓ)::AΓ:(AΓ + lr(AΓ)), or NΓ:AΓ::AΓ:(AΓ + lr(AΓ)). Therefore, sq.AΓ = rect.NΓ,(AΓ + lr(AΓ)).
2) From 1), sq.AΓ:rect.ΓN,MΞ is the same as rect.NΓ,(AΓ + lr(AΓ)):rect.NΓ,MΞ, which is the same as (AΓ + lr(AΓ)):MΞ. Now, VII.19 states that sq.AΓ:(sq.BK + sq.lr(BK)):: rect.ΓN,MΞ: (sq.NM + sq.MΞ), or, sq.AΓ:rect.ΓN,MΞ::sq.BK + sq.lr(BK)::sq.NM + sq.MΞ. From here, then, Halley can conclude that (AΓ + lr(AΓ)):MΞ::sq.BK + sq.lr(BK)::sq.NM + sq.MΞ.

sq.lr(BK)] to the sum of the axis and its *latus rectum*, the rectangle contained by MΞ and a length, arising from the application, which is ψ, will be equal to the sum of the squares of MN and MΞ.[32]

Now, the axis of the section will be either greater than its *latus rectum*, or less, or equal to the same; so first let it be greater than it; hence (by VII.6), MΞ will be greater than NM.[33] But (by Elem.II.7) the squares of NM,MΞ together will equal twice the rectangle contained by NM,MΞ and the square of NΞ: for this reason, the rectangle contained by MΞ and the given line ψ, just found, will equal twice the rectangle NM,MΞ and the square of NΞ. But NM is the excess of MΞ over NΞ; and so twice the rectangle NM, MΞ is equal to twice the excess of the square of MΞ over the rectangle NΞ,ΞM: therefore, the square of NΞ together with twice the excess of the sq.MΞ over the rectangle NΞ,ΞM equals the rectangle: and removing from both twice this excess, the difference that the rectangle MΞ, ψ has over twice the excess of the sq.MΞ over the rectangle MΞ,ΞM will be equal to the square of NΞ: and, halving, the rectangle contained by MΞ and 1/2ψ together with NΞ [*viz.* the rectangle MΞ,(1/2ψ + NΞ)] less the square on MΞ will equal half the square of NΞ.[34] But the square of NΞ is given: therefore, the rectangle contained by MΞ and 1/2ψ together with NΞ less the square on MΞ is given: but it [MΞ,(1/2ψ + NΞ)] adjoins a given line, 1/2ψ and NΞ taken together of course; therefore, MΞ is given; and the point Ξ being given, so too will the point M be given.

Following the synthesis, which we shall omit, Halley adds the *diorismos*:

Nor is there any limit other than what is demonstrated in VII.44 that the sum of the squares of the axis and its *latus rectum* is less than the sum of squares of the sides of the figure for any other diameter. In this way, therefore, if the proposed sum will be less, the point M falls beyond the vertex or outside the section . . . and, hence, the problem will not be possible.

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[32] Since (AΓ + lr(AΓ)):MΞ::sq.BK + sq.lr(BK)::sq.NM + sq.MΞ, also, rect.(AΓ + lr(AΓ)), ψ:rect.MΞ, ψ::sq.BK + sq.lr(BK): sq.NM + sq.MΞ. Therefore, since ψ is defined so that rect.(AΓ + lr(AΓ)),ψ = sq.BK + sq.lr(BK), it follows that rect.MΞ, ψ = sq.NM + sq.MΞ.

[33] Halley also needs the fifth corollary following VII.31, which says that if the axis of an hyperbola is greater than its *latus rectum* then any other transverse diameter will be greater than its *latus rectum*. Thus, BK > lr(BK). Then by proposition VII.6, BK:lr(BK)::MΞ:NM, and it follows that MΞ > NM.

[34] From above, we have rect.MΞ, ψ = sq.NM + sq.MΞ.

By *Elem.*II.7, sq.NM + sq.MΞ = 2rect.NM,MΞ + sq.NΞ.

But 2rect.NM,MΞ = 2rect.(MΞ − NΞ),MΞ = 2(sq.MΞ − rect.NΞ,MΞ).

So, rect.MΞ,ψ = sq.NΞ + 2(sq.MΞ − rect.NΞ,MΞ), or 1/2rect.MΞ,ψ − sq.MΞ + rect.NΞ,MΞ) = 1/2sq.NΞ.

From which we have, rect.(MΞ,1/2ψ + NΞ) − sq.MΞ = 1/2sq.NΞ.

Now, proposition VII.44 in its entirety states:

> If there is a hyperbola, and the transverse side of the figure constructed on its axis is either
>
> 1) not less than its *latus rectum*, or
> 2) less than it, but [such that] its square is not less than half of the square of the difference between it [the transverse side] and it [the *latus rectum*]:
>
> then the sum of the squares of the two sides of the figure constructed on the axis is less than [the sum of] the squares of the two sides of any figure constructed on one of its other diameters.

Proposition VII.44 covers only the first case; VII.45 covers the second. Similarly, Halley's proposition VIII.29 is built around the first case of VII.44, while his proposition VIII.30, which states "With the same things set out, let the axis of the hyperbola now be less than its *latus rectum*; and so it is necessary to find the diameter of the section, which has the squares of the sides of the figure taken together equal to a given rectangle," is built around the second. Halley is, indeed, following Book VII very closely—and not only here. His proposition VIII.26 sets the problem: given the axis and *latus rectum* of an ellipse, to find a diameter that together with its *latus rectum* makes a given sum. This is connected to VII.41 which tells us that in an ellipse the sum of the sides of the figure constructed on any diameter is greater than the sum of the sides of figure constructed on the major axis and less than that on the minor axis. Halley's VIII.27 sets the problem: given the sides of the figure constructed on the axis of an hyperbola, to find a diameter whose figure is equal to a given rectangle, and his VIII.28 proposes the same problem for the ellipse. These are connected to VII.42,43 which tell us that the figure constructed on the axis of an hyperbola or an ellipse (*i.e.*, the figure constructed on the *major* axis of the ellipse) is smaller than the figure constructed on any other diameter (and in the ellipse less than that constructed on the minor axis).

A detailed discussion of Halley's production of his edition of the *Conica* and of his reconstruction of Book VIII is beyond the scope of this book; and whether or not Halley's work is, *in fact*, close to Apollonius's original is also not the purpose of the present discussion. What we have shown from this brief look at Halley's reconstruction is only that 1) a master geometer, like Halley, could find problems for which the propositions in Book VII are "of great use," and 2) a

reader of Halley's Book VIII could hold open Apollonius's Book VII and see how the propositions in that book could, by suggesting *diorismoi*, be the foundation of problems such as those in Halley's work.

The Diorismic Theorems of Book VII

Even if it were proved conclusively that Halley's reconstruction of Book VIII was right on the mark, that it hardly differed from Apollonius's own Book VIII, this still would not prove that Book VII exists only for the sake of Book VIII. Rather, one would have to view the relationship between the propositions in these books as being like that between Euclid's *Elem.*VI.27 and *Elem.*VI.28 discussed above; put side by side, it is likely that one would be able to see how propositions in Book VII anticipate problems in Book VIII. But what we stressed about Euclid's *Elem.*VI.27 and *Elem.*VI.28 was that while VI.27 is used in an essential way in VI.28 it should not be seen as a mere lemma; *Elem.*VI.27 has a certain independence, an independence which makes it a theorem needing no further justification. Indeed, need it be said that something can have value in its own right and yet serve for another thing? In this way, although the theorems in Book VII may be easily adaptable for the *diorismoi* of problems and may even be useful in suggesting the problems themselves, they are still theorems that prove "many wonderful and beautiful things on the topic of diameters and the figures constructed on them," as Apollonius says in the introduction to Book VII. Let us review the contents of some of these theorems.

Book VII can be divided roughly into three parts: part 1 from proposition VII.1 to VII.20; part 2 from VII.21 to VII.31; part 3 from VII.32 to the end. In part 1, we find a series of proposition that appear to be lemmas for the other proposition in the book. But, as we shall see later in some detail, a truer view of these propositions is that they provide the *framework* for thinking about diameters and their *latera recta*; in particular, they develop the crucial device of Book VII, the *homologue.* The propositions in part 2 concern, for the most part, diameters and their conjugates alone, while those in part 3 concern diameters, their *latera recta*, and the figures constructed from these.

Although, as we said, we shall discuss the theorems in part 1 below, for the present discussion it is important to point out that

two propositions there, propositions VII.12,13, must be counted among those "wonderful and beautiful things" about diameters that Apollonius mentions in his introductory letter. Proposition VII.12 states that "[In] any ellipse, the sum of the squares on any two of its conjugate diameters whatever is equal to the sum of the squares on its two axes," while VII.13 states the related result for the hyperbola, namely, that "[In] every hyperbola, the difference between the squares on its axes is equal to the difference between the squares on any pair of its other conjugate diameters whatever." These two propositions are used later in the book and may well have been useful in the problems of Book VIII; indeed, Halley made good use of them in his reconstruction of Book VIII. They stand out because of their generality and because they are stated only in terms of the diameters themselves, rather than in terms of any auxiliary lines. But they stand out too in that they do not appear to be *diorismic theorems*. The same could be said, moreover, about one of the striking theorems of the second part of Book VII, in fact, the proposition *concluding* that part, proposition VII.31. That proposition states: "When a pair of conjugate diameters is drawn in an ellipse or between conjugate opposite sections . . ., then the parallelogram bounded by that pair of diameters with angles equal to the angles formed by the diameter at the center is equal to the rectangle bounded by the two axes," or, as is actually proven, the parallelograms formed by the tangents at the ends of conjugate diameters are all equal.[35] This proposition too might have been useful in formulating problems of the kind appearing in Book VIII,[36] but, like VII.12,13, it is not of the form that would make it useful in suggesting *diorismoi*. It does not refer to things being least or greatest, that is, it does not provide any *limit* for the givens of a problem, unless it be a trivial limit, namely, that the sum or difference of squares on the conjugate diameters, or the parallelogram formed by the same must *be equal* to some quantity.

These propositions, therefore, might appear out of place in a book that Apollonius says concerns diorismic theorems. But this does not

[35] It is not entirely clear why Apollonius states the proposition as he does instead of in terms of the tangents, which Toomer says with some reason is "easier to visualize" (*Apollonius* Conics *Books V to VII*, p. lxxvi). It is likely, however, that he wanted the subject of the proposition to be figures whose sides are conjugate diameters *themselves* rather than any other lines associated with them. After all, VII.31 is the final proposition in a series of propositions concerned with the conjugate diameters.

[36] Halley made use of it in two problems, his VIII.17 and 18.

mean that diorismic theorems in and of themselves are the subject of the book. The *subject* of Book VII is diameters, conjugate diameters, *latera recta*, and the figures constructed on diameters, and propositions such as VII.12,13,31 (and to a lesser extent VII.29,30)[37] underline that fact. And in this connection it is worth observing too that Apollonius concludes the fifteen summarizing corollaries (which are, for the most part, diorismic in form) following VII.31 by writing only that "These are matters which can be proven from what we proved *in the treatment of the diameters and figures of the sections, and their [the figures'] sides, and the ratios of the conjugate diameters and latera recta* [emphasis added]*." Furthermore, while the vast majority of propositions in Book VII are, truly, diorismic theorems, this does not mean that such propositions do not provide positive information about diameters, conjugate diameters, and the rest. They do, and they also contain ancillary results which, taken by themselves, would not be seen as leading to *diorismoi.* Thus, in part 2 of Book VII, we have propositions, such as VII.21, that says that "If there is a hyperbola, and its transverse axis is greater than its erect axis, then the transverse diameter of each pair of conjugate diameters among its other diameters is greater than the erect diameter of [that pair]; and the ratio of the greater axis to the lesser axis is greater than the ratio of the transverse diameter to the erect diameter among the other conjugate diameters; and the ratio of a transverse diameter nearer to the greater axis to the erect diameter conjugate with it is greater than the ratio of a transverse diameter farther [from that axis] to the erect diameter conjugate with it." So, if the transverse axis of an hyperbola is ΑΓ and the conjugate axis is ΙΟ, and if ΑΓ > ΙΟ, then of all pairs of conjugate diameters the ratio ΑΓ:ΙΟ is

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[37] Proposition VII.29 tells us that "The difference between the figures constructed on [each of] the diameters of any hyperbola and [each of] the squares on those diameters are equal," while VII.30 tells us that "If there is added to [one of] the figures constructed on any of the diameters of an ellipse the square of that diameter, [the sum always] comes out equal." Proposition VII.29 follows easily from the definition of the second diameter following I.16 and VII.13, while VII.30 follows just as easily from I.15 and VII.12. Consider for example, VII.30. Let BK and TY be two diameters and let ZH and OI be their respective conjugates. Then VII.13 says that sq.BK + sq.ZH = sq.TY + sq.OI. But I.15 says that sq.ZH = rect.BK,lr(BK) and sq.OI = rect.TY,lr(TY). Therefore, sq.BK + rect.BK,lr(BK) = sq.TY + rect.TY,lr(TY), which is the claim of VII.30. Because VII.29 and VII.30 follow so easily from VII.12,13, and I.15, and the definition of the second diameter following I.16, we tend to see them as somewhat less striking than VII.12,13, and 31.

the greatest, and it is this statement that makes VII.21 a diorismic theorem. But the proposition also says that if ΑΓ > ΙΟ, then also any other transverse diameter will be greater than its conjugate, and that if ΒΚ and ΖΗ are transverse diameters, and ΒΚ is closer to ΑΓ than ΖΗ, then ΒΚ:conjugate (ΒΚ) > ΖΗ:conjugate(ΖΗ), and these clauses do *not* appear to be related to *diorismoi.* Yet statements like them, especially the latter, that premised on the relative closeness of ΒΚ and ΖΗ to ΑΓ, are found throughout the diorismic theorems of Book VII. And this, of course, is true not only in part 2 but also in part 3, where a typical proposition is VII.43: "The smallest of the figures constructed on diameters of an ellipse is the figure constructed on the major axis; and the greatest of them is the one constructed on the minor axis; and those constructed on [diameters] closer to the major axis are smaller than those constructed on [diameters] farther [from it]."

The point we want to emphasize is this. Book VII does, as Apollonius says in the general introduction, concern diorismic theorems, but diorismic theorems *are theorems*; we learn from them definite things about definite objects, in this case, about diameters, conjugate diameters, *latera recta*, and figures. While they have a certain *form*, their objects never lose their priority. Their importance and relevance must be judged by the depth of *what they say about* diameters, conjugate diameters, *latera recta*, and figures, and not just by their *use* in the construction of problems. Otherwise, they would, indeed, be mere lemmas.

At the start of this chapter we said that there is no inconsistency in Apollonius's referring to Book VII as a book concerning diorismic theorems and, at the same time, as one containing ". . . many wonderful and beautiful things on the topic of diameters and the figures constructed on them." What we have said until now explains what it means to refer to the book in this way: that Book VII is a book concerning diameters and the figures constructed on them, whose propositions, for the most part, are of a certain *type*, namely, they are diorismic theorems. But we still have not answered why Book VII *should be* referred to in this way. Again, the problem is that Book VII does not appear to have a monopoly on diorismic theorems. In particular, Book V also seems to contain propositions of this type. For example, here is V.64: "If a point is marked below the axis of a parabola or hyperbola, such that the line drawn from it to the vertex of the section forms with the axis an acute angle,

and [such that] it is not possible to draw from that point to the section a line such that the part of it falling between the section and the axis is one of the *minima*; or if only one of the lines drawn from that point to one side [of the axis], which is different from the side on which the point is, can have cut off from it [between axis and section] a minimum line: then the line drawn from that point to the vertex of the section is the shortest of the lines drawn from that point to that side of the section, and, of the remaining lines, those closer to it [the first line] are shorter than those farther [from it]." The structure of this enunciation is very similar to that of the enunciations in Book VII, such as VII.43 quoted above, and it is rather typical of the enunciations in Book V; it does seem to have the form of a diorismic theorem.[38] So, why does Apollonius refer only to Book VII as being a book of diorismic theorems? Why does he not also refer to Book V as a book of diorismic theorems containing beautiful things about maximum and minimum lines? The answer is not altogether clear. But we think it has to do with the specific subject of Book VII: diameters, conjugate diameters, *latera recta*, and the figures constructed on diameters.

These objects are of incontrovertible importance throughout the *Conica*, and they are the objects most likely to be connected with the givens of a problem involving conic sections. This is because the diameter is produced together with the conic sections themselves, as we saw in our chapter on the elements of conic sections, because the *latus rectum* is the vital component in the *symptoma* of each conic section, and because the ratio between the diameter and the *latus rectum*, which are the sides of the figure, governs innumerable properties of conic sections. The conjugate diameter is important, among other

[38] Knorr observes something along these lines regarding the propositions just preceding V.64. He writes "An interesting feature of Apollonius' formal style throughout this section on the construction of normals to the conics is that although these are clearly a set of *problems* of construction, they are presented in the form of *theorems* of *possibility* of construction, sc. 'it is possible to draw lines . . .'; hence, as *diorisms*. In standard usage among ancient writers, a diorism is that section of a theorem or problem establishing the conditions under which solution is possible . . . Diverging from this usage, Apollonius appears to cast the entire problem of construction in the form of a diorism when, as here in V,51,52 it depends on such a discriminating condition. Thus, when he says that his theorems are useful for 'the synthesis of *diorisms*' (prefaces to Books I and IV), he would seem to refer to *problems* of this sort" (*Ancient Tradition of Geometric Problems*, pp. 335–336, n. 73). We should add that from what has been said so far about diorismic theorems, such theorems take in a wider class of propositions than just those having words "it is possible . . ."

things, because the square on it equals the figure on the diameter and because it bears the same ratio to the diameter, in square, as does the *latus rectum* to the diameter. For if D is the diameter, d its conjugate, and lr(D) the *latus rectum* relative to D, then, by I.15 (for the ellipse) and the definition of the second diameter following I.16 (for the hyperbola), we have D:d::d:lr(D), so that sq.d = rect.D,lr(D) and sq.D:sq.d::D:lr(D).[39] In the last chapter, moreover, we saw the importance of the figure in determining when two ellipses or hyperbolas have the same shape, namely, that they will have the same shape if the figures constructed on the axes are similar. And this means they will have the same shape if the rectangles contained by the axis and its conjugate (or the parallelograms contained by conjugate diameters) are similar. For if D and d are the axes of one ellipse or hyperbola and D* and d* of another, then D:lr(D)::D*:lr(D*), so that the figures will be similar, if and only if sq.D:sq.d::sq.D*:sq.d* or D:d::D*:d*. Needless to say, these mutual relationships between conjugate diameters and their respective *latera recta* come up quite often in Book VII, with Apollonius mentioning I.15 or 16 explicitly in about a fifth of the propositions.

One must also consider what knowledge we have of diameters at the end of Book I. Besides knowing that every conic section has a diameter and that the diameter of an ellipse and hyperbola has connected with it a second diameter, we know that the second diameter and the principal diameter (that produced with the section) have the relationship set out in I.15,16, and that every section has as many diameters as there are lines cutting the section and, in the case of the parabola, parallel to the principal diameter, or, in the case of the ellipse and hyperbola, passing through the center. We also know how to find the *latus rectum* related to any of these other diameters given the principal diameter. Book II shows further when certain lines will be diameters and how to find diameters and axes. What else needs to be known? What knowledge does Book VII need to supply? What seems to be missing is the comparison of diameters—some ordering among all these diameters, a kind of τάξις.

Apollonius does appear to assume at least one ordering of diameters, namely, that in an hyperbola diameters closer to the axis are smaller than those farther away. He says this explicitly in VII.28

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[39] Or, alternatively, sq.D = rect.d,lr(d) and sq.d:sq.D::d:lr(d).

and uses this unproven fact several times before and after VII.28. For example, in VII.25 he proves that "[In] every hyperbola, the line equal to [the sum of] its two axes is less than the line equal to [the sum of] any other pair whatever of its conjugate diameters; and the line equal to the sum of a transverse diameter closer to the greater axis plus its conjugate diameter is less than the line equal to the sum of a transverse diameter farther from the greater axis plus its conjugate diameter." Given a hyperbola with ΑΓ as its axis (see fig. 5), Θ its center, and KB,ZH and OI,YT being two pairs of conjugate diameters, Apollonius has to prove that ΑΓ and its conjugate is less than KB + ZH and YT + OI, and that KB + ZH is less than YT + OI. In the first part of the proof, he supposes that the axes are equal, and thus, by VII.23, also KB = ZH and YT = OI. Then he simply says "But diameter KB is greater than axis ΓΑ, and diameter YT [is greater] than diameter KB. Thus what we desired has been proven." It does follow from V.34 that the axis is the shortest diameter of an hyperbola and that other diameters are longer the farther they are from the axis,[40] and it is proven explicitly in V.11 that the minor axis is the shortest diameter in an ellipse and that other diameters are longer the closer they are to the major axis. But we tend to agree with Toomer, who says "Presumably it [the fact regarding the hyperbola] was considered self-evident."[41] This is because:

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[40] Proposition V.34 states "If a point is marked outside a conic section on a maximum or minimum line extended, then the smallest length intercepted between that point and the section (on lines drawn from that point on either side of the section [*sic*] but not extended to cut the section in more than one point) is the line which is the extension of the maximum or minimum line; and of the other lines, those closer to it are smaller than those farther [from it]." The result then follows from the fact that a line along the axis from a point whose distance to the vertex equals half the *latus rectum* is a minimum line (by V.6).

[41] Toomer, *Apollonius* Conics *Books V to VII*, p. 603. Toomer adds, "However, if this is so, one should not need to prove the proposition separately, and by a different method, when the axes are unequal," which Apollonius does. This worry is answerable in two ways: 1) to apply the fact that diameters are longer the farther they are from the axis for the conjugate diameters would involve bringing in the conjugate sections, and this Apollonius seems to avoid throughout the later books; 2) the different method to which Toomer refers involves using VII.13 and, accordingly, one of the important theorems Apollonius has just proven. We ought to say a few more words about this second answer. With ΝΞ as the conjugate axis to ΑΓ (in Apollonius's text this conjugate axis is left unnamed), Apollonius says that since, by VII.13, sq.ΑΓ – sq.ΝΞ = sq.BK – sq.ZH, therefore, ΑΓ + ΝΞ < ZH + KB. Apollonius says this, then, as if it is an easy and completely obvious step. This implies that he must have had a very clear picture in his mind. In the case ΑΓ > ΝΞ (the case

1) Apollonius never cites the relevant propositions from Book V, nor, for that matter, any others from Book V, and 2) Apollonius never relies on results from any but the books of elements, Books I to IV, in the later books of the *Conica*, Books V to VII.[42]

If indeed Apollonius took this basic ordering of the diameters in the ellipse and hyperbola to be self-evident for any reader of the book, then that might have served as the model for the general direction of the propositions in the book. In other words, once one has in mind that the axis is the smallest diameter (or largest, in the case of the major axis of an ellipse) and other diameters are longer the farther they are from the axis (or shorter in the case of the major axis of an ellipse), then it becomes clear that the next step is to ask about the *latera recta* of the axis as compared to the other diameters, the conjugate diameters (keeping in mind the relationships among diameters, *latera recta*, and conjugate diameters discussed above), sums of conjugate diameters, sums of squares of conjugate diameters, and so on. Such ordering by size, therefore, suggests from the start a diorismic form as a natural form for the propositions of the book. For, as we have suggested, in speaking of limits—of the greatest and the least—diorismic theorems are also ordering theorems. So, this

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where $A\Gamma < N\Xi$ is similar), this picture, we believe, is that of the two equal gnomons given by the two differences of squares sq.$A\Gamma$ – sq.$N\Xi$ and sq.BK – sq.ZH (fig. 6). Recalling that sq.$A\Gamma$ – sq.$N\Xi$ = rect.($A\Gamma + N\Xi$),($A\Gamma - N\Xi$) and sq.BK – sq.ZH = rect.($BK + ZH$),($BK - ZH$) and that $A\Gamma < BK$ (this is assumed in any case), the picture becomes almost a "proof without words." If this indeed is what Apollonius had in mind, then the obviousness of the proof gives some support to our supposition that Apollonius wanted, in this instance, to illustrate the significance of VII.13. Toomer does not, apparently, give weight to the kind of visual intuition that makes this part of VII.25 obvious. He gives a much less straightforward argument that, in fact, is erroneous. Following Ver Eecke (who is generally dependable in such matters) Toomer says "This is not a trivial consequence," and then argues as follows: Since sq.$A\Gamma$ – sq.$N\Xi$ = sq.BK – sq.ZH, we have rect.($A\Gamma + N\Xi$),($A\Gamma - N\Xi$) = rect.($BK + ZH$),($BK - ZH$). The result will then follow if it can be shown that ($A\Gamma - N\Xi$) > ($BK - ZH$) (this is also the crux of our "proof without words"). Suppose that $A\Gamma > N\Xi$ (again, the case where $N\Xi > A\Gamma$ should be argued similarly), then VII.21 provides that $A\Gamma:N\Xi > KB:ZH$ so that also $A\Gamma:(A\Gamma - N\Xi) < KB:(KB - ZH)$. But $A\Gamma < KB$. Therefore, Toomer and Ver Eecke argue, $(A\Gamma - N\Xi) > (BK - ZH)$. But this does *not* follow, just as, reproducing the steps of the argument numerically, it does not follow from $10 < 15$ and $10:(10-5) < 15:(15-9)$ (that is, from the fact that $10:5 > 15:9$) that $(10-5) > (15-9)$!

[42] Two exceptions in two directions ought to be mentioned. First, as we observed in our chapter on Book IV, Apollonius does not use any of the results from Book IV in any other of the extant books. Second, results from Book VII, of course, were presumably used in Book VIII, as we discussed above.

together with the fact that what are to be ordered are things having to do directly with diameters, *latera recta*, and conjugate diameters, whose crucial importance is undeniable in the basic understanding of conic sections and, therefore, also in problems concerning conic sections where diorismoi would be found, we believe, is a strong reason why Book VII is singled out as concerning diorismic theorems. But now we need to turn to the meat of the book. We need to look at the procedure by which Apollonius approaches his demonstrations in Book VII, or as we put it above, at the *framework* for his thinking about diameters, *latera recta*, conjugate diameters, and figures.

Images of Diameters and Latera Recta

As in proposition I.11 for the case of the parabola, in propositions I.12,13, the *latus rectum*, the ὄρθια, of a hyperbola and ellipse is defined at the same time these sections themselves are defined, and, like them, the *latus rectum* is defined in terms of the cone. For the hyperbola, it is defined as a line ZΛ (see fig. 7) such that the transverse diameter ΘZ has the same ratio to ZΛ that the square on AK, the line from the vertex to the base of the cone drawn parallel to the diameter, has to the rectangle contained by BK and KΓ, the segments AK cuts off from the diameter of the base circle. Its use is given in the *symptoma* of the hyperbola, namely, that if MN is an ordinate, that is, if MN is a line drawn from the section to the diameter parallel to the common section of the cutting plane and the base of the cone, then the square on MN is equal to the parallelogram ". . . applied to ZΛ, having ZN as breadth, and exceeding by a figure ΛΞ similar to the rectangle contained by ΘZ and ZΛ." Quickly, almost imperceptibly, the rectangle contained by the transverse diameter ΘZ and the *latus rectum* ZΛ becomes the "figure (εἶδος)" of the hyperbola.[43] Thus, in I.14 Apollonius refers to the transverse

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[43] The formal definition of the figure is given finally in Book VI (definition 10): "The figure which I call the *figure of the section constructed on the axis or on the diameter* is that [figure] contained by the axis or diameter together with the *latus rectum*." The fact that Apollonius already uses the word "figure" in this sense in I.14 makes it plausible that the definition in Book VI is a later interpolation. If true, this could be interpreted to mean that, for Apollonius, the upright side and transverse side *as the sides of a rectangle*, the "figure," is something so natural as to warrant no special mention.

side as "... the transverse side *of the figure* (τοῦ εἴδους ἡ πλαγία πλεύρα) [emphasis added], that between the vertices of the sections...," or in I.21, with AB being the diameter and AΓ the *latus rectum*, Apollonius says, "... let the straight line BΓ *determining the figure* [emphasis added] be joined." It is as the upright side of the "figure" that the *latus rectum* finds its main concrete geometric manifestation. Indeed, its name, "upright side (ὄρθια)," has little sense and its being perpendicular to the diameter, its very "uprightness," has little use *unless* it is taken as the side of a rectangle, as we have said before.

The *ratio* of the sides of the "figure" is a ratio that comes up one way or another all through the *Conica*. For example, in V.9 we are shown that if EΓ along the axis of a hyperbola is greater than half the *latus rectum* of the hyperbola (see fig. 8), and HZ:ZE is the same ratio as the ratio of transverse diameter to the *latus rectum*, where H is the center of the hyperbola, then if ZΘ is perpendicular to EΓ, EΘ will be a minimum line. In this example, as in others like it, one might view ZE as the *counterpart* of the *latus rectum* whose visible significance is in the "figure".[44] But does *Apollonius* intend us to think of the geometrically cognizable "figure" when we are faced with a proportion such as that in V.9? The *homologue* in Book VII provides, we think, at least one instance where he clearly does.

The *homologue* makes its first appearance in VII.2, 3 (VII.2 in the case of the hyperbola and VII.3 in the case of the ellipse) and,

[44] We are influenced here indirectly by K. Saito's views in his "Book II of Euclid's *Elements* in the Light of the Theory of Conic Sections." Saito shows there that "[The theory of proportions and the use of propositions in *Elem*.II] are not methods of treating the lines and areas as general quantities in a way similar to modern algebra, but they are the means for transforming between 'visible' and 'invisible' forms of areas" (p. 43). Accordingly, Saito says that "The alleged algebraic methods, *i.e.*, 'geometric algebra' and the theory of proportions, which have long been assumed to be the backbone of Apollonius's thought, bear only peripheral importance to the course of the investigations of new results. Their role can be summed up as follows:

1) To transform known or desired relations into a form suitable for the use of geometric intuition;

2) To prove the relation already anticipated by geometric intuition;

3) To transform the result into a form suitable for formal statements as a proposition" (p. 46).

Saito's examples from the *Conica* come entirely from Books I–III, and this is completely in line with what we said in the first part of this chapter concerning the character and place of the early books in the whole of the *Conica*. The interplay between the "visible" and "invisible" realms in the later books is less the *subject* of the inquiry, but is nevertheless, we claim, still present.

thereafter, is used widely in Book VII.[45] Proposition VII.2 states that "If the axis in a hyperbola is extended in a straight line so that the part of it falling outside of the section is the transverse diameter, and a line is cut off adjoining one of the ends of the transverse diameter such that the transverse diameter is divided into two parts in the ratio of the transverse diameter to the *latus rectum*, while the line cut off corresponds to the *latus rectum*, and a line is drawn from that end of the transverse diameter which is the end of the line which was cut off to the section, in any position, and, from the place where [that line] meets it [the section], a perpendicular is drawn to the axis, then the ratio of the square on the line drawn from the end of the transverse diameter to the rectangle contained by the two lines between the foot of the perpendicular and the two ends of the line which was cut off is equal to the ratio of the transverse diameter to the excess of it [the transverse diameter] over the line which was cut off. And let the line which was cut off be called the 'homologue'".

Here is the proof:

> Let the hyperbola be the section with extended axis ΑΓΕ [fig. 9], and let the figure of the section be ΓΔ. Let line ΑΘ be cut off from line ΑΓ, and let
>
> ΓΘ:ΘΑ = ΓΑ:ΑΔ (which is the *latus rectum*).
>
> We draw from point Α to the section an arbitrary line, ΑΒ, and draw line ΒΕ perpendicular to the axis.
>
> Then I say that sq.ΑΒ:rect.ΘΕ,ΕΑ = ΑΓ:ΓΘ.[46]
>
> Proof of that:
>
> We make rect.ΑΕ,ΕΖ = sq.ΒΕ.
> Therefore, rect.ΑΕ,ΕΖ:rect.ΑΕ,ΕΓ = sq.ΒΕ:rect.ΑΕ,ΕΓ.
>
> But {sq.ΒΕ:rect.ΑΕ,ΕΓ}[47] equals the ratio of the *latus rectum* (which is ΑΔ) to the transverse diameter (which is ΑΓ),

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[45] It is worth noting, with Toomer, that "No text containing this term [*i.e.*, the "*homologue* of the figure"] survives in the original Greek, but Halley conjectured that Apollonius had adapted the word ὁμόλογος to this special meaning, and translated accordingly" (*Apollonius* Conics *Books V to VII*, p. 595).

[46] As in the previous chapters, we shall write 'Therefore', 'is similar to', 'rect.AB,CD', and 'sq.AB', where Toomer has '∴', '|||', '(AB·CD)', and 'AB^2' respectively.

[47] These are Toomer's brackets. As for their significance, Toomer explains, "Parentheses (like this) are used, not for additions by me, but either to make the

as is proven in Prop.21 of Book I.[48]

Therefore rect.AE,EZ:rect.AE,EΓ = ΔA:AΓ = ZE:EΓ,

and ΔA:AΓ = AΘ:ΘΓ.

Therefore, ZE:EΓ = AΘ:ΘΓ.

Therefore, ZΓ:ΓE = AΓ:ΓΘ.

Therefore, ZA:ΘE = AΓ:ΓΘ.

But, when we make AE a common height,

ZA:ΘE = rect.ZA,AE:rect.EΘ,EA.

Therefore AΓ:ΓΘ = rect.ZA,AE:rect.AE,EΘ.

But rect.ZA,AE = sq.AB.[49]

Therefore, sq.AB:rect.AE,EΘ = AΓ:ΓΘ.

The *homologue* is the line AΘ, since it has been constructed by dividing the transverse axis AΓ at a point Θ such that ΓΘ:ΘA::transverse axis: *latus rectum*; it is a fixed line, since it is produced by dividing another fixed line, the transverse axis, in a fixed ratio. This first main point about the *homologue*, that it is a fixed line for any given hyperbola (or ellipse), goes hand in hand with the second main point about the *homologue*, that it has a name. Plainly, a line such as ZE in V.9 does not have a special name because it is *not* fixed—ZE is determined only to the extent that E is determined, and, except for E being farther from the vertex than half the *latus rectum*, E is arbitrary. Having a name, and, accordingly, a definition, puts the *homologue* on a par with the other three line segments in the *Conica* that also have a definition, and which are intimately associated with the equally defined conic sections: the transverse diameter, the upright

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structure of the thought clearer, or to enclose parts of mathematical expressions, for which I also employ { }." (Toomer, *op. cit.*, p. xci.)

[48] There is a certain sense in which VII.2,3 not only rely on I.21, but at the same time complement it. This can be seen best by considering I.21 and VII.2,3 in general terms. Thus, in I.21 the square on a line drawn from the section ordinatewise to the diameter of the section has to a rectangle, the difference (or *sum* in the case of an ellipse) of whose sides is a fixed line (the transverse side), the same ratio that the *latus rectum* has to the transverse side, while in VII.2 (and VII.3) the square of a line drawn from the section to the *vertex* of the section has to a rectangle, the difference of whose sides is a fixed line (the *homologue*), the same ratio that the transverse side has to the excess of the transverse side over the *homologue* (in the case of the ellipse, to the transverse side *together* with the *homologue*), or, in terms of the *latus rectum*, the ratio of the transverse side together with the *latus rectum* (in the case of the ellipse, the *excess* of the transverse side over the *latus rectum*) to the transverse side. We should add that these formulations are, as Taisbak has pointed out (in his "Theorem of Conjugate Application of Areas"), equivalent to the hyperbolic and elliptic application of area formulations one finds in the *symptomata* of the hyperbola and ellipse.

[49] ZA = AE + EZ, so rect.ZA,AE = sq.AE + rect.EZ,EA. But Z has been placed so that rect.EZ,EA = sq.BE, therefore rect.ZA,AE = sq.AE + sq.EB = sq.AB.

diameter, and the upright side or *latus rectum*. Obviously, ΘΑ has the *particular* name, *homologue*, because in the proportion, ΓΘ:ΘΑ::transverse axis: *latus rectum*, ΘΑ *corresponds* to the *latus rectum*, and is, therefore, in this sense, *homologous* to the *latus rectum*. It also plays an analogous part to the *latus rectum* of the parabola in VII.1, whose main purpose is, precisely, to be the analogous counterpart to VII.2,3.[50] For in VII.1, Apollonius proves that "If the axis of a parabola is extended in a straight line outside of the section to a point such that the part of it which falls outside of the section is equal to the *latus rectum*, and furthermore a line is drawn from the vertex of the section to any point on the section, and a perpendicular to the axis drawn from where it meets it [the section], then the line which was drawn [from the vertex] is equal in square to the rectangle contained by [1] the line between the foot of the perpendicular and the vertex of the section, and [2] the line between the foot of the perpendicular and the point to which the axis was extended." In particular, if AB is a parabola (see fig. 10), ΑΓ its axis, ΑΔ its *latus rectum*, then with AB being any line from the vertex to the section, VII.1 shows that sq.AB = rect.ΔΓ,ΑΓ. So, to make the comparison, in VII.1 the ratio of the square on AB to a rectangle whose sides differ by the *latus rectum* is always the same (namely, the ratio borne by two equal magnitudes), while in VII.2,3 the ratio of the square on AB to a rectangle whose sides differ by the *homologue* is always the same. That we are to see the *homologue* as analogous to the *latus rectum*, not only as a line having a given ratio to the transverse diameter, but also *as the upright side of the "figure"* is clear from the proof of VII.2. For in the *ekthesis*, Apollonius presents us with the "figure" ΓΔ explicitly (and, with that, the diagrams accompanying VII.2,3 have drawn in them the *whole rectangle* whose sides are the transverse diameter and *latus rectum*, that is, *the whole "figure"*), rather than only with the *latus rectum*, ΑΔ. We know that ΑΔ *is* the *latus rectum*, of course, because we know *where* it is in the "figure." The *homologue* is

[50] We completely concur with Toomer when he writes "Presumably VII.1 (which is not used in Book VII, although it may have been used in Book VIII) is inserted to show the analogy with the propositions for ellipse and hyperbola in VII.2–3" (*Apollonius* Conics *Books V to VII*, p. 595). Regarding the possible application of VII.1 in Book VIII, Halley, at least, uses VII.1 nowhere in his reconstruction of that book, whereas he *does* use VII.2,3 (in VIII.5,6, for example).

then defined in terms of the parts of this "figure": "ΓΘ:ΘΑ = ΓΑ:ΑΔ (which is the *latus rectum*)," and, therefore, the *homologue* is set in direct comparison to the *latus rectum* as the upright side of the "figure." This comparison is clearly the principal reason Apollonius presents us with the "figure," for in the proof ΓΔ is *not* used *as a figure*, as opposed to the use of the "figure" as a figure in, say, VI.12 where it is shown that "Hyperbolas and ellipses in which the figures constructed on their axes are similar are also [themselves] similar; and if the sections are similar, then the figures constructed on their axes are similar."

So, the *homologue* is, first of all, analogous to the *latus rectum* in the strict sense that it corresponds to the *latus rectum* in a proportion, an *analogia*. But it is also analogous to the *latus rectum* in the sense that the *homologue* is assimilated to the *latus rectum*. Four factors work together to give us this view of the *homologue*: 1) that it is a fixed line, given the hyperbola (or ellipse) and its transverse axis; 2) that it has a name; 3) that it functions in VII.2,3 in a way analogous to the *latus rectum* of the parabola in VII.1; 4) that it is put in direct comparison with the *latus rectum* in the latter's fullest geometrical manifestation, namely, as the upright side of the "figure." The four observations above have nothing to do with the logical demands of the book; the validity of the proof of VII.2,3 is neither strengthened nor weakened by Apollonius's calling ΑΘ the *homologue*. These other aspects of the *homologue* give us, rather, *a way of thinking* of this line, ΑΘ, of relating it to the geometrical significance of the *latus rectum*.

It hardly needs to be said, however, that the proofs of the propositions where the *homologue* appears *require* only that the *homologue* is analogous to the *latus rectum in the first strict sense*. But here too we can point to a kind of image-making capacity. Up until now, we have concentrated on the fixed line ΑΘ, but, of course, also ΓΘ, which is the image of the transverse diameter, is fixed as well. The single segment ΑΓ, which is the transverse diameter, divided into the two contiguous segments ΓΘ and ΘΑ, provides us ever with a concrete image of the ratio between the transverse axis and its *latus rectum*. One tends to forget that a strictly mathematical analogy, a proportion, is also a matter of image-making. One tends to forget that because a mathematical analogy is also a *precise* image, and, therefore, one that can be used in proof. Yet, surely it is one of the hallmarks of Greek mathematics, in particular, that its mathematical

statements could have potency both in bringing forth concrete images and in providing stepping stones in exact demonstrations.

That Apollonius was truly interested in having the reader *see* the ratio of the transverse axis to its *latus rectum* in fixed contiguous concrete lines in the diagram, as well as other important ratios, can be inferred from the next proposition concerning the parabola, VII.5. In this connection, it is worth noting that Book VII contains only three propositions directed to the parabola, VII.1,5,32. These propositions seem to be included mostly as introducing propositions on the hyperbola and ellipse, the chief concern of the book, since it is they which contain conjugate diameters and figures. We have already seen how this is true regarding VII.1. It is clear also regarding VII.32, which states that the *latus rectum* with respect to the axis of the parabola is the smallest among all the *latera recta* in a parabola and those related to diameters closer to the axis are smaller than those farther away, and which, accordingly, introduces the sequence of propositions concerning the relative sizes of the *latera recta* of hyperbolas. Proposition VII.5, on the face of it, does not fit this pattern. It states, "If there is a parabola, and one of its diameters is drawn in it, and from the vertex of that diameter a perpendicular is drawn to the axis, then the parameter of the lines drawn from the section to the diameter parallel to the tangent drawn from the vertex of the diameter, [i.e.], when figures are applied to that line, [those ordinates] are equal in square to them—and that line is its [the diameter's] *latus rectum*—is equal to the *latus rectum* of the axis increased by four times the amount cut off from it by the perpendicular from the axis adjacent to the vertex of the section." Toomer remarks that "This is a lemma for Prop.32, which is the only place where it is used,"[51] and, on first sight, this seems reasonable since, unlike VII.1 and VII.32, it is hard to connect the theme of VII.5 with any other proposition following it. On the other hand, if it is meant to be only a lemma to VII.32 why does it appear so early? We think that its purpose is to introduce VII.6,7 in which one sees how the homologue is used to image the ratio of the squares of conjugate diameters, and, therefore, also the ratio of any transverse diameter to its *latus rectum.*[52]

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[51] Toomer, *op. cit.*, p. lxxi.

[52] See our discussion in the previous section on the use, in this regard, of I.15 and the definition following I.16 of the second diameter.

The evidence for this is in the mode Apollonius chooses to prove the proposition.

With Book I at hand, it is not at all hard to construct a fairly straightforward proof for VII.5: Let AB be the parabola, and let BI be a diameter (see fig. 11). Let AH be its axis, and AΓ be its *latus rectum* with respect to AH. Draw BZ from B perpendicular to AH. Then we need to show that the *latus rectum* with respect to BI (for which, as before, we shall write lr(BI)) equals AΓ increased by a line four times AZ. Draw EA perpendicular to the axis, extend line BI to E, and draw tangent BΔ to the section at B. Let BΔ intersect AZ extended at Δ and AE at Θ. Then by I.49, ΘB:BE::ΔB:ΔZ::lr(BI):2ΔB. Therefore, 2sq.ΔB = rect.ΔZ,lr(BI) = 2rect.AZ,lr(BI), since BΔ is tangent to the parabola. So, rect.AZ,lr(BI) = sq.ΔB, from which we have, since sq.ΔB = sq.BZ + sq.ΔZ = sq.BZ + 4sq.AZ, sq.BZ = rect.AZ,lr(BI) – 4sq.AZ = rect.AZ,(lr(BI) – 4AZ), taking AZ as a common height. But, sq.BZ = rect.AΓ,AZ. Therefore, lr(BI) – 4AZ = AΓ, or lr(BI) = AΓ + 4AZ, Q. E. D.

But this is *not* Apollonius's proof. *His* proof runs as follows:

> We draw line EA perpendicular to the axis, and extend line IB to E and draw line BΔ tangent to the section at point B, and draw line BH so that it forms a right angle with BΔ (see fig. 12).
>
> Then ΔBΔH is similar to ΔBΘE.
> Therefore, BΘ:BE = ΔH:BΔ.
>
> Therefore line ΔH is equal to half the *latus rectum* to diameter BI, as is proven in Prop.49 of Book I.
>
> But rect.ΔZ,ZH = sq.BZ,
> because < ΔBH is right and line BZ is perpendicular [to ΔH].
> And sq.BZ = rect.AΓ,AZ.
> Therefore, rect.ΔZ,ZH = rect.AΓ,AZ.
> But ΔZ = 2AZ, as is proven in Prop.35 of Book I.
> Therefore, AΓ = 2ZH.
> And 4AZ = 2ΔZ.
> Therefore, AΓ + 4AZ = 2ΔH.
>
> And we have [already] shown that 2ΔH is the *latus rectum* to diameter BI.
>
> Therefore the *latus rectum* to diameter BI is equal to line AΓ plus four times AZ.

Apollonius's proof of VII.5 is not shorter than ours, and, requiring an extra construction line BH, it does not strike one as being more elegant. So, why does he proceed this way? The key, we believe,

lies precisely in that extra construction line. Because of the line BH, there is a crucial difference between the demonstrations. In ours, the *latus rectum* with respect to BI *does not appear in the diagram*, and the *latus rectum* with respect to the axis appears *only* because we drew it! In Apollonius's proof, by contrast, the *half latera recta*, both with respect to BI and the axis, appear explicitly as line segments along the axis, as ΔH and ZH, respectively. Thus, the ratio of these two segments, obviously, is the same as the ratio between the *latera recta* themselves, and they are placed in such a way, namely, being along the axis and having the same endpoint, so that that ratio is easy to see. This is where VII.5 can be seen to be a preparation for VII.6,7; in those propositions too a pair of segments, having, in this case, the same ratio as a diameter to its *latus rectum*, are produced so that they both lie along the axis of an hyperbola and an ellipse and have the same endpoint, so that again the ratio has a concrete representation that is easy to see, a tangible image. The construction given in VII.6,7 is repeated over and over throughout the book, and, in this sense, presents the basic pattern for Apollonius's investigation. So, let us now turn to that construction.

Proposition VII.6 states: "If there are constructed on the extension of the axis of a hyperbola two lines adjacent to the two ends of the axis which is the transverse diameter, each of them being equal to the line which we called 'homologue', and placed as it [the homologue] is placed, and two conjugate diameters from among the diameters of the section are drawn, and, from the vertex of the section, a line is drawn parallel to the 'erect' diameter of the two to cut the section, and from the place where it intersects it a perpendicular is drawn to the axis: then the ratio of the transverse of the two conjugate diameters to the erect one is equal in square to the ratio of the line between the foot of the perpendicular and the end of the more remote of the two homologues to the line between the foot of the perpendicular and the end of the nearer of the two homologues; and the ratio of the transverse diameter to the parameter of the lines drawn to it [the transverse diameter] parallel to the second diameter, which [parameter] is its *latus rectum*, is, in length, equal to the ratio of the two lines which we mentioned previously to each other in length." So, let the transverse axis of the hyperbola be AΓ with center Θ (see fig. 13), AN, ΓΞ be the homologues (*i.e.*, ΓN:NA:: transverse:latus rectum::AΞ:ΞΓ), and ZH and BK be conjugate diameters (the latter being the transverse diameter and the former the

"erect" diameter). Then, if line ΛΛ is drawn parallel to line ZH,[53] and ΛM is drawn perpendicular to the axis, Apollonius proves that sq.BK:sq.ZH::ΞM:MN and, from that, that BK:lr(BK) is also the same ratio as ΞM:MN.[54] Proposition VII.7 is the corresponding proposition for the ellipse, and the construction is identical except that the points Ξ and N, there, lie on the extension of the axis (see fig. 14).[55]

We shall omit the proofs of VII.6,7. However, we must point out that what is really proved in both cases is the first claim, that sq.BK:sq.ZH is the same ratio as ΞM:MN, whereas the second claim, that BK:lr(BK) is also the same ratio as ΞM:MN, is stated but hardly argued, nor does it have to be.[56] Why does Apollonius emphasize such an obvious conclusion? It is because the goal of the propositions is to explain the significance of the line segments ΞM and MN, namely, that they provide a derivative image of a diameter and its conjugate, being in the same ratio as that between the conjugate diameters *in square*, and that they provide an image of a diameter and its *latus rectum*. This may be why Apollonius remarks at the end of VII.7, as an illustration, that from what was proven in the proposition, ". . . it will be proven that, if the perpendicular falling from point Λ on the axis passes through center Θ, then diameter KB will be equal to diameter ZH, because MΞ = MN." That one can learn to look at the diagrams in VII.6,7 and *see* the significance of MΞ and MN, allows Apollonius to take as obvious, for example, the twelfth corollary in the "summary" following VII.31: "It has also been shown that . . . [in] every ellipse, the transverse diameter of the figures of the section constructed on the diameters drawn between

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[53] There are two diagrams, one for the case where the *latus rectum* is shorter than the transverse axis, and one for the case where the *latus rectum* is longer than the transverse axis, but the proof, however, is the same for both. Our diagrams (as well as those for VII.7) are simplified to show only the details of the construction; the complete diagram is the same as our fig. 15 without the segment ΞI.

[54] Although this is mentioned in the enunciation it is *not* restated in the *diorismos*. Toomer writes that "This may have fallen out by haplography" (*op. cit.*, p. 597), which is a reasonable suggestion considering that in VII.7, where the same proposition is proved for the ellipse, the analogous statement *does* appear in the definition of what is to be proved.

[55] If one thinks of the axis, however, as one continuous line, with one part within the section (in the case of the hyperbola, within the opposite sections) and one part outside the section, then, in both cases, Ξ and N are placed in that part of the axis which is outside the section, so, in this light, the correspondence becomes somewhat closer.

[56] In VII.6, the text cites, II.1,21 two propositions to justify the step from

the major axis and the two equal conjugate diameters is greater than their [corresponding] latus rectum." The corollary is obvious because, for any diameter drawn between the major axis and the two equal conjugate diameters, the point M falls between Θ and A (see upper diagram in fig. 14). Let us then review what images, or, since they are such because of their place in various proportions, what *analogues* the construction contains.

First of all, ΓΞ and AN, the *homologues*, are each the analogue for the *latus rectum* of the figure on the axis, and AΞ and ΓN are each the analogue for the axis. Alternatively, ΓΞ and AN can be taken as analogues for the second axis, and AΞ and ΓN for the *latus rectum* of the figure on that axis.[57] This is because, by I.15 and the definition of the second diameter after I.16, it follows that if D is a diameter and d its conjugate then D:lr(D)::lr(d):d,[58] so that, ΓN:NA::AΞ:ΞΓ::D:lr(D)::lr(d):d. In the hyperbola, AΓ is the analogue for the sum of the axis and its *latus rectum*, and NΞ is the analogue for the difference between the axis and its *latus rectum.* In the ellipse, it is the other way around, that is, NΞ is the analogue for the *sum* of the axis and its *latus rectum*, while AΓ is the analogue for the *difference* between the axis and its *latus rectum.* These analogues are, of course, fixed for any given hyperbola or ellipse. For any given diameter in an hyperbola or ellipse, MΞ is the analogue of the diameter and MN of its *latus rectum*, as we saw above. In three propositions, propositions VII.8,9,10, there appears another line segment drawn along the axis, ΞI, which is the mean proportional between MΞ and NM, and so it can be taken as an analogue to the conjugate diameter, since the conjugate diameter is the mean proportional between the diameter and its *latus rectum.*

How far these analogies go can be seen in the propositions VII.1–20. Indeed, the diagrams for these propositions, lettering included, are virtually the same as those in VII.6,7, suggesting that the very pur-

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sq.BK:sq.ZH::ΞM:MN, to BK:lr(BK)::ΞM:MN, instead of the definition of the second diameter following I.16. Since, no justification is given at all in VII.7 for the corresponding step, the erroneous justification in VII.6 appears to be the work of an interpolator, and it may be assumed that in Apollonius's original text no justification appeared at all.

[57] This alternative is indeed used by Apollonius in VII.37,41,47.

[58] This is so since, using the same notation, *both* D:d::d:lr(D) and d:D::D:lr(d) are true by I.15 and the definition of a second diameter after I.16.

pose of propositions VII.1–20, with the possible exceptions of VII.12,13, is the development of these basic analogies. Consider, for example, VII.8. Following on the heels of VII.7, it says "Furthermore, we set the diagram for hyperbola . . . and ellipse . . . in the way it was in Prop.6 and Prop.7 of this Book [see fig. 15]*:[59] then I say that the ratio of the square on ΑΓ (which is the transverse diameter) to the square on BK and ZH, the two conjugate [diameters], when they are joined together in a straight line, is equal to the ratio of rect.ΝΓ,ΜΞ to the square on a line equal to line ΜΞ plus the line which is equal in square to rect.MN,ΜΞ)." The line equal in square to rectangle contained by MN and ΜΞ is, of course, ΞΙ, which, as we said, corresponds to the conjugate diameter, so that the correspondence of the second and fourth terms in the proportion proved is clear: the second term is the square on the line comprising the diameter and conjugate diameter, BK and ZH, while the fourth term is the square on the line comprising the lines corresponding to these, ΜΞ and ΞΙ, respectively. But what about the first and third terms? Establishing *that* correspondence is, in a way, the point of the demonstration. Here then is how Apollonius proves VII.8:

We make line ΞΙ a mean proportional between lines NM, ΜΞ.
Then sq.ΑΓ:sq.BK = sq.ΑΘ:sq.ΘΒ.
But sq.ΑΘ = rect.ΔΘ,ΘΕ, as is proven in Prop.37 and Prop.38 of Book I.
Therefore, sq.ΑΓ:sq.BK = rect.ΔΘ,ΘΕ:sq.ΘΒ.
But rect.ΔΘ,ΘΕ:sq.ΘΒ = rect.ΑΓ,ΓΜ:sq.ΓΛ, because lines ΔΒ,ΒΘ are parallel to lines ΑΛ,ΛΓ [respectively].
Therefore, rect.ΔΘ,ΘΕ:sq.ΘΒ [*sic*][60] = sq.ΑΓ:sq.BK.
And when we make ΓΜ a common height,
ΓΛ:ΓΝ = rect.ΑΓ,ΓΜ:rect.ΓΜ,ΓΝ.
And sq.ΓΛ:rect.ΞΜ,ΜΓ = ΑΓ:ΑΞ,
as is proven in Prop.2 and Prop.3 of this Book.
And ΓΝ = ΑΞ, because lines AN and ΓΞ are the two homologues.
Therefore, rect.ΑΓ,ΓΜ:rect.ΓΜ,ΓΝ = sq.ΓΛ:rect.ΞΜ,ΜΓ.
So, *permutando*, rect.ΑΓ,ΓΜ:sq.ΓΛ = rect.ΓΜ,ΓΝ:rect.ΞΜ,ΜΓ.
But we have [already] proven that
rect.ΑΓ,ΓΜ:sq.ΓΛ:sq.ΑΓ:sq.BK.

[59] In VII.6,7, the line ΓΛ is joined, BE is drawn perpendicular to the axis, and the line ΒΔ is drawn parallel to ZH. Line ΒΔ, therefore, is tangent to the section at B. It is shown, moreover, that ΓΛ is parallel to ΒΘ, since ΑΟ = ΟΛ (ΑΛ was drawn parallel to ZH and, accordingly, ordinatewise with respect to BK. Therefore, BK bisects ΑΛ at O) and ΓΘ = ΘΑ (Θ being the center of the section).

[60] As Toomer rightly points out, "rect.ΑΓ,ΓΜ:sq.ΓΛ" is meant.

Therefore, sq.AΓ:sq.BK = rect.NΓ,ΓM:rect.ΞM,MΓ = NΓ:ΞM.
And NΓ:ΞM = rect.NΓ,ΞM:sq.MΞ.
Therefore, sq.AΓ:sq.BK = rect.NΓ,ΞM:sq.MΞ.

Furthermore, sq.BK:sq.ZH = ΞM:MN,
as was proven in the two preceding propositions.
Therefore, BK:ZH = MΞ:ΞI,
because ΞI is mean proportional between ΞM and MN.
Therefore BK:(BK + ZH) = MΞ:MI.
Therefore sq.BK:sq.(BK + ZH) = sq.MΞ:sq.MI.
But we have [already] proven that
sq.AΓ:sq.BK = rect.NΓ,ΞM:sq.MΞ.
So, *ex aequali*, sq.AΓ:sq.(BK + ZH) = rect.ΓN,ΞM:sq.MI.
And MI equals MΞ plus the line whose square is equal to
rect.NM,MΞ, *etc.*

Clearly, the work of the proof is in the demonstration of the proportion, the *analogia*, sq.AΓ:sq.BK::NΓ:ΞM, and, from that, the proportion, sq.AΓ:sq.BK::rect.NΓ,ΞM:sq.MΞ. For the reader attentive to the analogies suggested by propositions VII.6,7, the result is expected to some degree since, as we have seen, NΓ can be related to the axis AΓ and ΞM to a given diameter, BK. What is being compared in the ratio, however, is the *square* on the axis and on a diameter different from the axis, not the axis and diameter. In effect, then, the second proportion presents the rect.NΓ,ΞM as the proper analogue for the sq.AΓ, because it is related to sq.MΞ which is the precise analogue for sq.BK. It ought to be observed, that following VII.8 the proportions, sq.AΓ:sq.BK::NΓ:ΞM and sq.AΓ:sq.BK::rect.NΓ,ΞM:sq.MΞ are used over and over, so that they must be viewed not only as the important conclusions for the proof of VII.8, but also as important conclusions of the proposition itself. The next important step in the proof is another analogy, which the reader already expects, namely, BK:ZH::MΞ:ΞI, for as we pointed out above, ΞI is the analogue for the conjugate diameter to BK. From here, the completion of the proof involves only rather routine manipulations of ratios.

If one sees such precise analogies as, in some sense, *abstract proportions* only, then it would surely be stating the obvious to say that the logic of the proof of VII.8 follows the logic of the analogies. But the proportions are *not* abstract: they relate lines along the axis to *specific lines of real significance in the conic section*, that is, to the axis, to diameters, to *latera recta*. The logic of analogies, the relating of these *specific* lines to their analogues, then, is what makes the proof *intelligible*, not just deductively correct. And it is what makes the *propositions*

themselves intelligible. Thus, VII.9 tells us with hardly any other description that, in an ellipse and hyperbola, sq.AΓ: sq.(BK – ZH) [or sq.(ZH – BK)]::rect.NΓ,MΞ:sq.(MΞ – ΞI) [or sq.(ΞI – MΞ)], in other words, the square on the axis is to the square on the difference between conjugates diameters as the *analogue* for square on the axis is to the square on the difference between the *analogues* for the conjugate diameters; VII.15 tells us that "the ratio of sq.AΓ to the square on the line which bounds, together with diameter BK, the figure of the section, which [line] is the *latus rectum* to diameter BK, equals {rect.NΓ, MΞ:sq.MN}," that is, the square on the axis is to the square on the *latus rectum* of the figure on another diameter as the *analogue* for the square on the axis is to the square on the *analogue* for the *latus rectum*; VII.20 tells us that sq.AΓ:(sq.BK – sq.T)::rect.NΓ,MΞ:(sq.ΞM – sq.MN), where T is the *latus rectum* of the figure on BK, that is, the square on the axis is to the difference between the square on a diameter and the square on its *latus rectum* as the *analogue* for the square on the axis is to the difference between the square on the *analogue* for the diameter and the square on the *analogue* for its *latus rectum.* All these propositions, then, relate significant lines in conics to other significant lines, their analogues, by means of pregnant analogies, proportions, the pregnancy involved being clearly geometrical, *i.e.*, visual and intuitive.

The ability of analogies to make the propositions intelligible is evident also in the analogies that are indirect, where lines along the axis are related to the specific lines in the section *in square only*; these are the analogies derived from the proportions sq.BK:sq.ZH::ΞM:MN and sq.AΓ:sq.BK::NΓ:ΞM. Thus, the lines ΞM, MN, and NΓ become related to the squares on BK, ZH and AΓ, that is, to the squares on conjugate diameters and on the axis, as their quasi-analogues.[61] So, for example, proposition VII.11 tells us, again without any further description except that we are to use the same diagram as in VII.6, that sq.AΓ:(sq.BK + sq.ZH)::ΓN:(NM + MΞ), in other words, that the square on the axis is to the sum of squares on a pair of conjugate diameters as the *quasi-analogue* to the square on the axis is to the sum of the *quasi-analogues* of the squares on the conjugate

[61] Of course, the lines cannot be compared *directly* to the squares, only in proportion; the various analogues for diameters are related to one another as the diameters are related to one another *in square only.*

diameters. It is worth noting that, although we do not know whether Apollonius applied VII.11in Book VIII,[62] *in Book VII*, the proposition is never used. Thus, one must ask, beyond how it might have been used, what it actually shows; the answer to that seems to be only that the relationship between the square on the axis and those on a pair of conjugate diameters is revealed in the relationship between the respective quasi-analogues, and manifesting this relationship seems to be the main point of Apollonius's proof. In other words, here too, as before, Apollonius seems to want not merely to establish a proportion but also to create an intelligible image of a relationship between intelligible lines.

Conclusion

In the first part of this chapter, we considered what Apollonius means by calling Book VII a book of diorismic theorems. We stressed that such theorems are theorems of a certain *type* rather than theorems directed to a particular *diorismos* in a particular problem; they have a form that makes them immediately adaptable to *diorismoi* while maintaining an independence from the problems themselves. This independence is what allows Apollonius to refer to the book as a book concerning diorismic theorems while referring to it in another place as a book containing beautiful results on the diameters and the figures constructed on them.

In the second part, we considered Apollonius's approach in this book of diorismic theorems, set out in the first twenty propositions of Book VII. Some measure of the distance between our account and interpretations current until now can be seen in Heath's discussion of what he calls the "Special use of auxiliary points in Book VII." There, Heath writes as follows:

> The seventh Book investigates the values of certain quadratic functions of the lengths of any two conjugate diameters PP′, DD′ in central conics of different excentricities [see our fig. 16], with particular

[62] Halley does make use of VII.11 in his VIII.15: "Given the axis and *latus rectum* of an hyperbola, find conjugate diameters of it in magnitude and position, for which the sum of squares is given" [*i.e.* that the sum of the squares on the conjugate diameters is given].

> reference to the *maximum* and *minimum* values of those functions. The whole procedure of Apollonius depends upon the reduction of the ratio $CP^2:CD^2$ to a ratio between straight lines MH′ and MH, where H, H′ are fixed points on the transverse axis of the hyperbola or on either axis of the ellipse, and M is a variable point on the same axis determined in a certain manner with reference to the position of the point P. The proposition that
>
> $$PP'^2:DD'^2 = MH':MH$$
>
> appears in VII. 6,7 . . ., and the remainder of the Book is a sufficient proof of the effectiveness of this formula as the geometrical substitute for algebraical operations.[63]

In our discussion, we also placed great weight on Apollonius's use of the *homologue* and of the proportion "$PP'^2:DD'^2 = MH':MH$" (or Apollonius's sq.BK:sq.ZH::MΞ:MN) set out in VII.6,7, but Heath sees the import of such things in a completely different light. First of all, Heath gives his greatest attention to the *points* H,H′, and M, rather than the *segments* AH, HA′, A′H′, H′A, MH′,MH.[64] Since the points determine the segments and the segments determine the points, one might view this as semantic quibbling on our part. It is not. Heath's thoughts on conic sections are led always by their equations, and the equations of curves relate the x and y coordinates of *points* on the curves when the curve is viewed in a coordinate system. By finding the fixed point H and the variable point M on the axis, Heath can see himself finding a fixed *parameter* and a *variable* x with which to set up a functional relationship with the lengths of the conjugate axes, determined by the coordinates of their endpoints. Accordingly, when Heath goes on to give an account of Apollonius's procedure in Book VII, what he shows is how to determine the points H,H′, and M on the axis, now taken essentially as the x-coordinate axis. He considers the case of the hyperbola having a and b as the lengths of its axes, and, hence, having the equation:

---

63 Heath, *Apollonius of Perga*, p. cxiii.

64 Toomer also refers to "homologue points," but he is at least careful to point out that "Strictly peaking, Apollonius never uses this term, but refers only to the 'homologue' *line*, which is a line, measured from one or the other end of the transverse axis to what I call the 'homologue points'. However, it is convenient to have a term to refer to these points, which are invariably marked with the letters N and Ξ in the main group of propositions in which the homologue is used" (*Apollonius* Conics *Books V to VII*, p. lxx, n. 2).

$$\frac{x^2}{\left(\frac{a}{2}\right)^2} - \frac{y^2}{\left(\frac{b}{2}\right)^2} = 1$$

Then, if C is the center, PP′ a diameter, and DD′ its conjugate he writes:

$$\frac{CP^2 + CD^2}{CP^2 - CD^2} = \frac{2(x^2 + y^2) - \left\{\left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right\}}{\left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2}$$

where x and y are the coordinates of P.[65] Of course, the expression relies on what *Apollonius* proves in VII.13, namely, that the difference between the squares on any diameter and its conjugate always equals the difference between the squares on the axes, though, in analytic geometry this can be shown without the machinery Apollonius provides in Book VII—and it is analytic geometry that is, admittedly, the source of Heath's development. After simplifying the expression above by means of the equation for the hyperbola, Heath obtains the expression:

$$\frac{CP^2 + CD^2}{CP^2 - CD^2} = \frac{4\left((2x^2 - \frac{a^2}{4}\right)}{a^2 \cdot \frac{a^2 - b^2}{a^2 + b^2}}$$

He then explains, "Apollonius' procedure is to take a certain fixed point H on the axis whose coordinates are (h,0), and a variable point M whose coordinates are (x′,0), such that the numerator and denominator of the last expression are respectively equal to 2ax′, 2ah . . ."[66] Thus, he obtains for the denominator, $\frac{h}{\left(\frac{a}{2}\right)} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right)$, from which it follows that,

[65] Note also that $CP^2 + CD^2 = 2CP^2 - (CP^2 - CD^2)$.

[66] Heath, *op. cit.*, p. cxiv.

$$\frac{\frac{a}{2} - h}{\frac{a}{2} + h} = \frac{b^2}{a^2}$$

or,

$$AH{:}A'H = b^2{:}a^2 = p_a{:}AA',$$

where, of course, p_a is the *latus rectum*, or parameter, relative to the axis.

"Thus," he says, "to find H, we have only to divide AA′ in the ratio p_a:AA′. This is what is done in VII.2,3 . . ."[67]

Heath's algebraic outlook leads him to investigate "Apollonius' procedure" in Book VII analytically, as a matter of finding the coordinates of points. In Apollonius's text there is, needless to say, no sign of any such analysis, nor is there any clear reason why even the kind of geometrical analysis possible in Greek mathematics should be necessary here. On the contrary, rather than seeing points H and H′ as points that ought to be derived, it appears evident that Apollonius simply divided a line that has obvious significance and in a ratio that has equally obvious significance as a suggestive *starting point.* Indeed, that lines *can be suggestive* of other lines whose significance is clear and concrete, was our guiding thought in thinking about the *homologue* and the other lines which reoccur over and over in Book VII. We have, accordingly looked at proportions in their image-making capacity, that is, in their capacity to bestow meaning on lines by making them into analogues for such objects as axes, diameters, *latera recta,* and figures on these. The proper use of proportions obviously makes a proposition deductively sound; we have emphasized here, in addition, that looking at what analogues the proportions produce also makes a proposition intelligible.

In the next chapter, we follow this thread further. There we shall again concentrate on images, but images that should be taken more as likenesses, images that are not precise, not linked by strict proportions. We believe, nevertheless, that such likenesses are also crucial in making the *Conica* an intelligible book.

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[67] Heath, *op. cit.,* p. cxv.

Figures

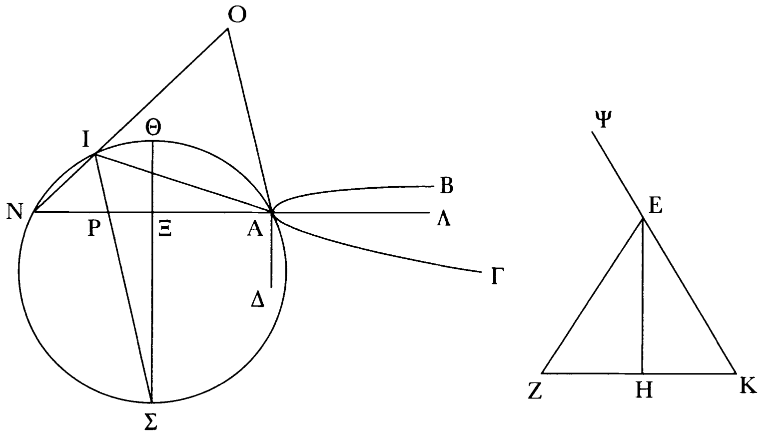

Fig. 1

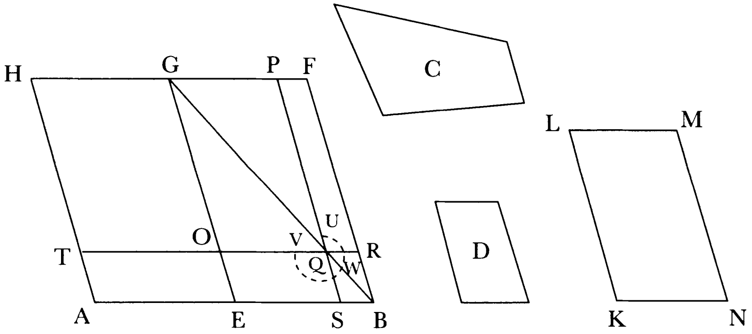

Fig. 2

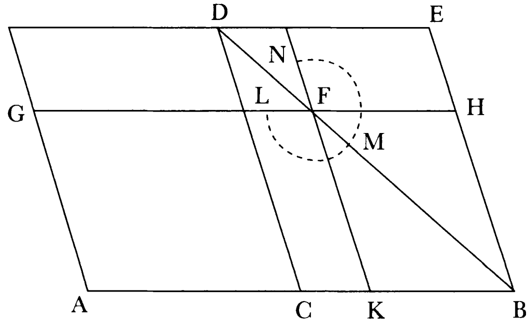

Fig. 3

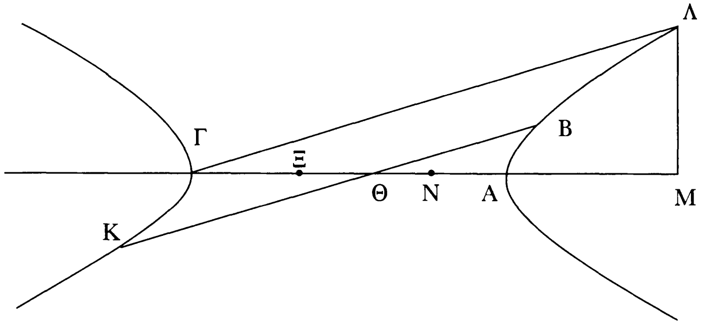

Fig. 4

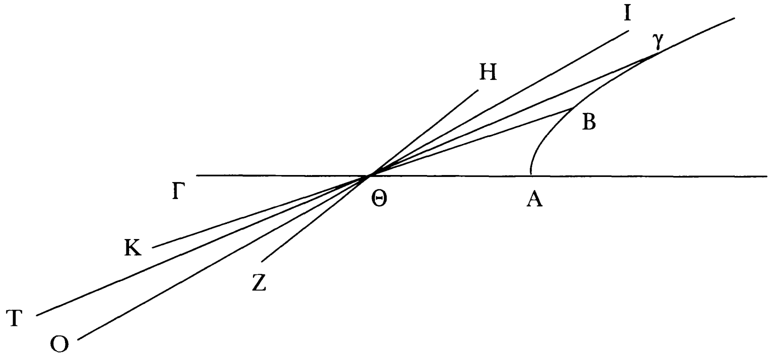

Fig. 5

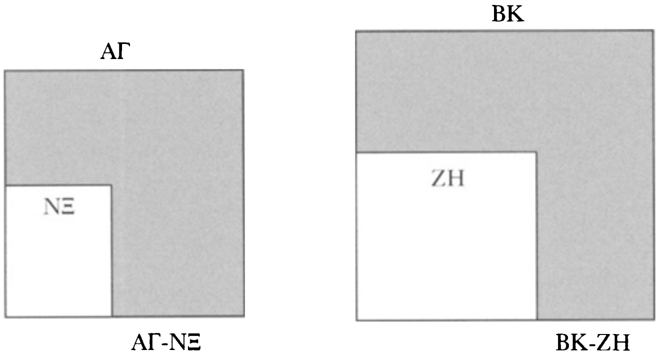

Fig. 6

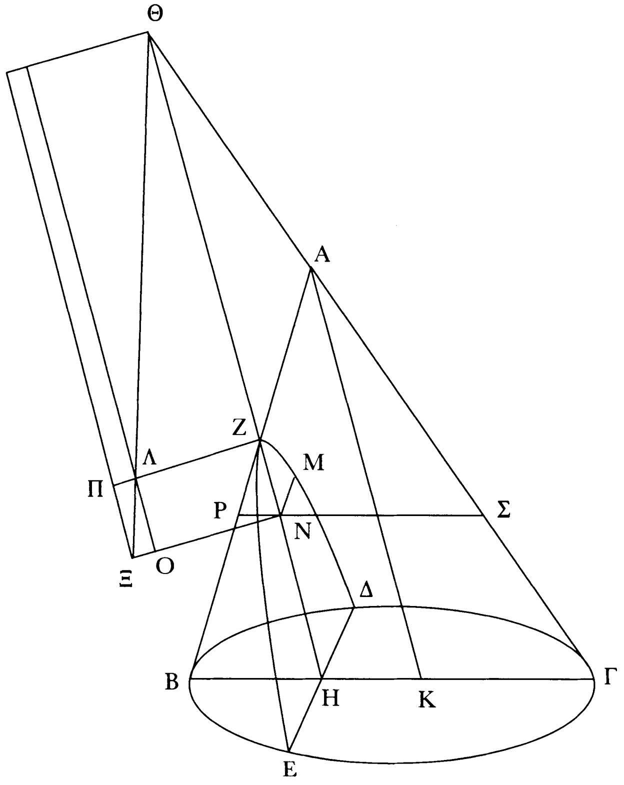

Fig. 7

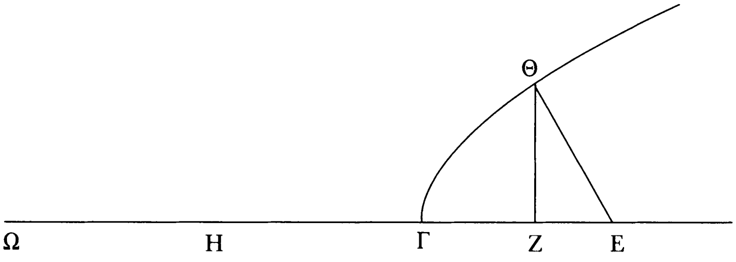

Fig. 8

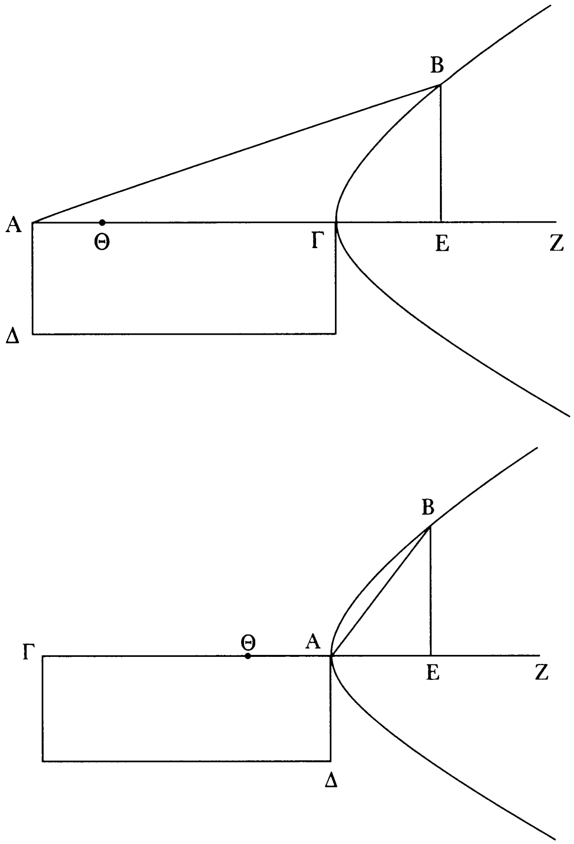

Fig. 9

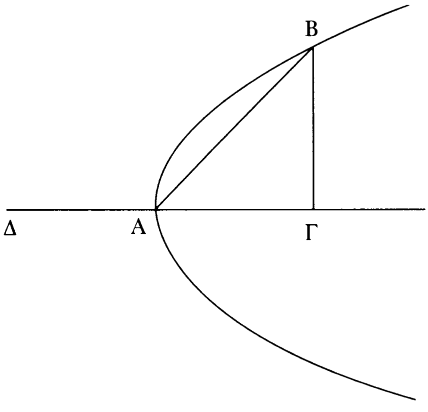

Fig. 10

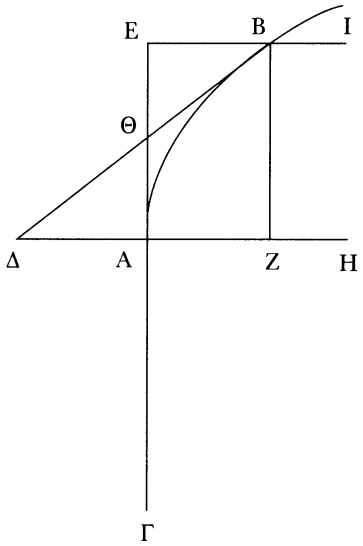

Fig. 11

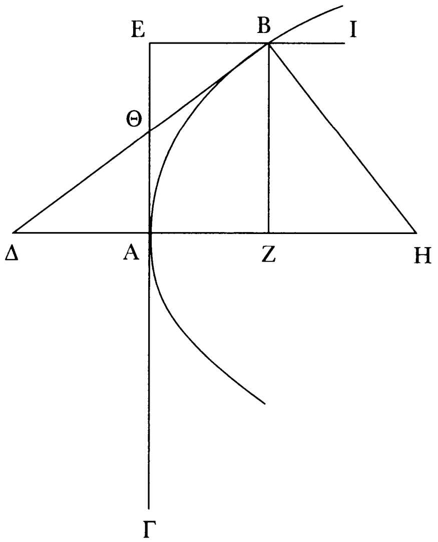

Fig. 12

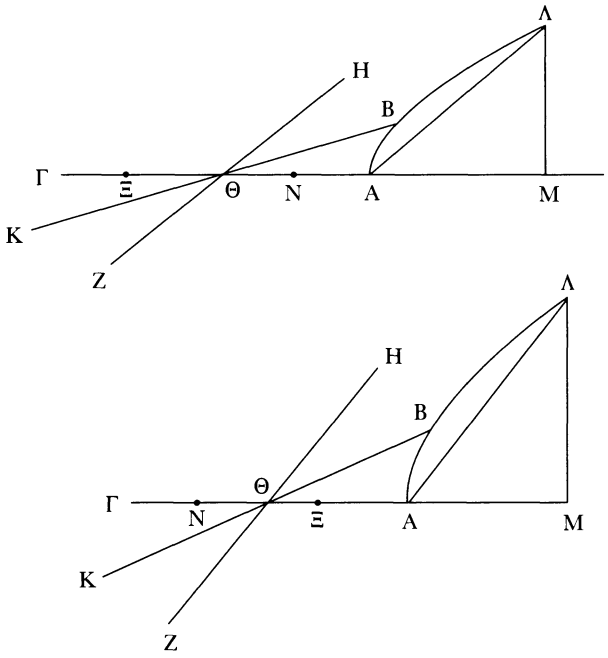

Fig. 13

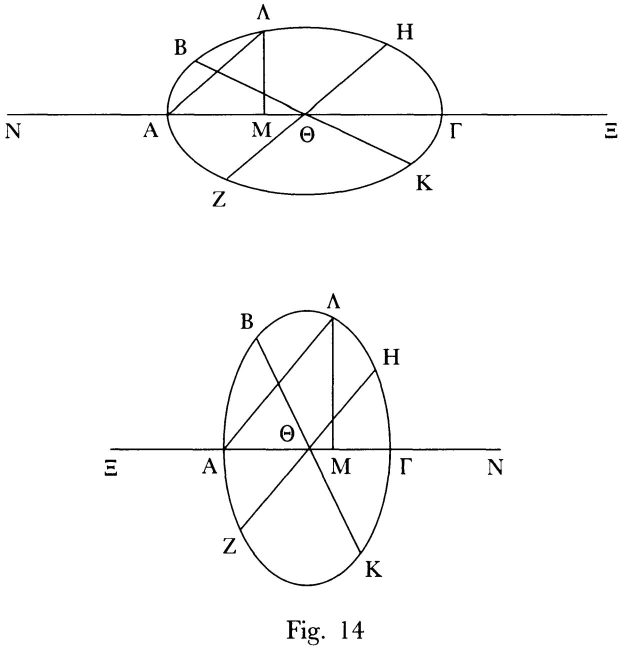

Fig. 14

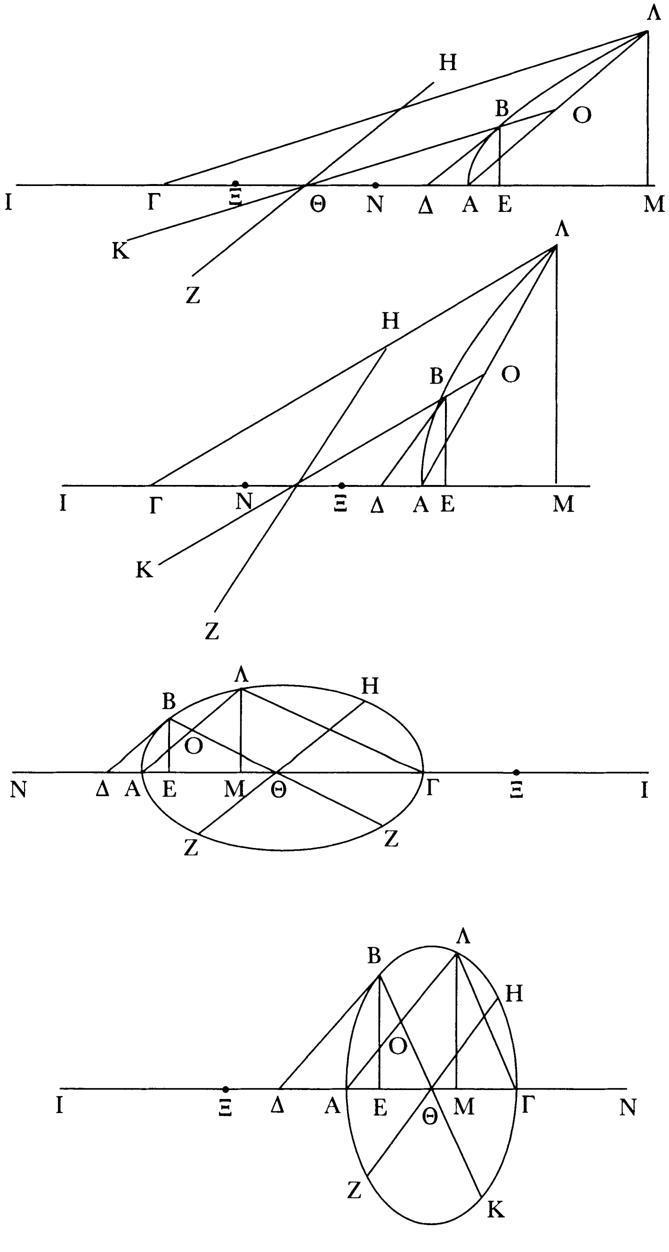
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Fig. 15

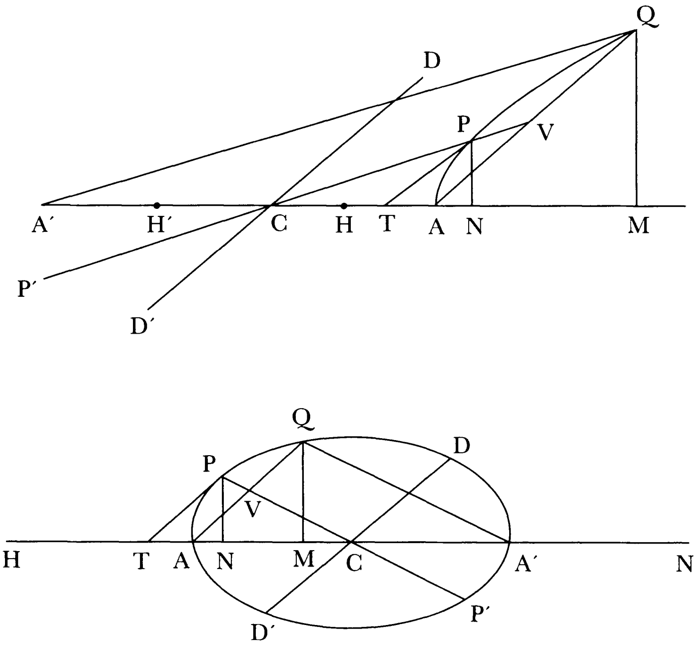

Fig. 16

CHAPTER SEVEN

ONCE AGAIN, THE USE OF ANALOGY IN APOLLONIUS: CONICS AND CIRCLES

Introduction

In the last chapter, we examined how Apollonius's employment of the *homologue* and other related lines in Book VII made the contents of that book intelligible and concrete. The key insight was that a proportion, an αναλογία, can also be an *analogy* in its usual sense, as something presenting an image or a likeness. In this chapter, we shall consider analogies once again, but this time analogies that must be understood as *only* presenting images, that is, where there is no *precise* relationship between object and image, no strict proportion.

We shall look at two examples. The first is the analogy between the tangents and asymptotes; it can be termed an *internal* analogy since it compares objects developed strictly within the confines of the *Conica*. The second is the analogy between circles and conic sections; it can be termed, by contrast, an *external* analogy since it involves analogies with ideas or works outside the *Conica*. The first example is important in that it allows us a glimpse into how Apollonius might have thought about one of the most novel and non-intuitive objects in the *Conica*, the asymptotes of the hyperbola. The second example, however, must clearly have the greater impact in the general interpretation of the *Conica* as a whole, and, accordingly, it is given more weight in the present chapter. Indeed, what we shall claim is that with the analogy between the circle and the conic sections comes the further analogy between *books* on conic sections and *books* on the circle such as Book III of Euclid's *Elements*. We shall suggest that, while Apollonius did not, necessarily, *base* his book on Book III of the *Elements*, books such as that must have guided him to a great extent in his thinking about conic sections and in his writing of the *Conica*. This means we can look to Euclid's book on circles as charting out the kind of subjects Apollonius would have thought appropriate to include in a systematic investigation of conic

sections. To this extent, the wholeness of the *Conica* becomes a reflection of the wholeness of Book III of the *Elements*.

To avoid any misunderstanding, we should say at the outset that we do not claim that 1) analogy has, for Apollonius, anything like the force of deduction, and that 2) analogy is the *only* other non-deductive element at work in giving shape to the whole of the *Conica*. The second point should already be clear from what we have said elsewhere in the present work—our chapter on the elements of conic sections, for example, showed the part played by visual intuitions in influencing the way Apollonius proceeds and links ideas in the work—while the *particular* importance of analogy will, we hope, become clear in the following pages. As for the first point, one should never forget that the *Conica is* a mathematical work; its assertions are followed by proof, and analogies are *not* proofs.[1] Yet, between the demands of proof and the whole structure of a work, its flow and development of ideas, its choice of things to be proved, there is much room for variation. The latter is, of course, the bread and butter of the interpreter. Indeed, specifically to the extent that things are *not* determined by purely logical considerations, they reveal more of the idiosyncrasy of Apollonius's thought.[2]

The importance of analogy in Greek thought has been noted by other writers.[3] In ancient literature, one finds Aristotle giving some

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[1] This means that, more often than not, one can only *infer* that analogies and the like are at work in the text; Apollonius does not make such thinking explicit. Indeed, the very fact that Archimedes, in the *Method*, *does* expose something of what lies behind his thinking is the excitement of that work—for such revelations are rare in the writings of Greek mathematicians. The commentators, on the other hand, do, naturally, try to account for their subject's thinking, and, later, we shall find some evidence in the writings of Eutocius for an analogy in the *Conica* between conic sections and the circle.

[2] This is, we think, true generally, but it is particularly true in the interpretation of Greek mathematics where we can say with Mueller that, ". . . the differences between Hilbert's and Euclid's geometry stem from a fundamental contrast between the dominant role of structure [*i.e.*, *purely logical* structure] in modern mathematics and its virtual absence in ancient mathematics" (*Philosophy of Mathematics and Deductive Structure in Euclid's* Elements, pp. ix–x).

[3] A couple of almost random examples from recent literature are William B. Boon, "Some Thoughts on Aristotle's Analogy Between Nature and Art" (*St. John's Review*, XLIV (1997), 51–72) and Indra Kagis McEwen, *Socrates' Ancestor* (The MIT Press, 1994), which considers the way architecture served as a continual analogy for Greek cosmological and philosophical thought. These examples, of course, show only the *plausibility* of analogy being also at work in Greek mathematical thought.

attention to the use of analogy-like arguments.[4] In his *Rhetorica*, for example, in the section on "paradigms" in rhetorical arguments, Aristotle speaks explicitly of the use of historical and illustrative parallels and fables.[5] In other places, such as in the *Posterior Analytics*, he refers to analogical arguments when he says that divisions may be made "... by analogy (κατὰ τὸ ἀνάλογον): for we cannot find a single identical name to give to a squid's pounce, a fish's spine, and an animal's bone, although these too possess common properties as if there were a single osseous nature."[6] And in the *Topics*, Aristotle speaks of the use of "similarity" (ὁμοιότης),[7] saying among other things that "The consideration of similarity is useful both for inductive arguments and for hypothetical reasoning and also for the assignment of definitions."[8] In modern times, the best known instance dealing with the topic is G. E. R. Lloyd's *Polarity and Analogy*. While Lloyd does not consider analogy as the only type of argumentation in Greek science, he does consider analogy one of the principal types.[9] Like us, Lloyd "... take[s] 'analogy' in its broadest sense, to refer not merely to proportional analogy (*a:b::c:d*) but to any mode of reasoning in which one object or complex of objects is likened or assimilated to another (of the two particular instances between which a resemblance is apprehended or suggested, one is generally unknown or incompletely known, while the other is, or assumed to be, better known) ..."[10] With these considerations in mind, let us then turn to our first example, the tangent and the asymptote.

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[4] See Lloyd, *Polarity and Analogy*, pp. 403–416, and Talyor, *Aristotle*, pp. 29–30.

[5] *Rhet.*B,1393a22–1394a18.

[6] *Post. Anal.*, 98a20–24. The English translation is G. R. G. Mure's. Aristotle mentions such arguing by analogy also at 76a38 and 99a15.

[7] *Topics*, 108a7ff. The connection between "similarity" or "likeness" and "analogy" is clear right from the start of Aristotle's discussion where he says "Likeness (ὁμοιότητα) must be viewed in things belonging to different genera, as one thing is to one thing so is another to an another (for example, as knowledge is to an object of knowledge so is perception to an object of perception) ..." (108a7–10).

[8] *Ibid.* 108b7–9. The English translation is by E. S. Forster.

[9] Lloyd, *Polarity and Analogy*, p. 7. The other important mode of argumentation that Lloyd considers is "polarity," regarding which he says, "Few of those who study early Greek thought can fail to be struck by the recurrent appeal to *pairs of opposites* of various sorts both in general cosmological doctrines and in accounts of particular natural phenomena" (*ibid.*).

[10] *Ibid.*, p. 175.

The Tangent and the Asymptote: An "Internal" Analogy in the Conica

Neither "tangent" nor "asymptote" in general are defined by Apollonius in the *Conica*, nor, to our knowledge, by any Greek mathematician. It is true that Euclid defines the tangent (ἡ ἐφάπτομενη) *to a circle* in *Elem.*III, def.2, saying that "A straight line is said to touch a circle (κύκλου ἐφάπτεσθαι) which, meeting the circle and being produced, does not cut the circle." Yet, the full force of this definition only comes through in the second part of *Elem.*III.16 where Euclid shows that ". . . into the space between the straight line [which is tangent to the circle] and the circumference another straight line cannot be interposed . . ." Apollonius also speaks of tangents being those falling outside the sections, such as in I.17,[11] whose proof Apollonius concludes saying, "Therefore the straight line drawn from the point A [the vertex of the given section] parallel to an ordinate will not fall within the line; therefore it will fall outside; and therefore it is tangent to the section." But, as in the case of Euclid's III.16,[12] the development of the notion of a tangent is incomplete without I.32, that "If a straight line is drawn through the vertex of a section of a cone, parallel to an ordinate, then it touches the section, *and another straight line will not fall into the space between the conic section and this straight line* [emphasis added]." Apollonius's proof of the second part (what is italicized above) recalls Euclid's proof of this fact for the circle, the former arriving at the contradiction that a greater line equals a lesser, while the latter arrives at the contradiction that a lesser line is greater than a greater. Thus, in *Elem.*III.16, having demonstrated that the straight line AE drawn at right angles to the diameter AB (see fig. 1) will fall outside the circle, Euclid continues by supposing that another line, FA, can be interposed into the space between the circumference CHA and AE. He then says, ". . . let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right,

and the angle DAG is less than a right angle,
AD is greater than DG.

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[11] I.17 states that "If in a section of a cone a straight line is drawn from the vertex of the line, and parallel to an ordinate, it will fall outside of the section."

[12] Cf. Dijksterhuis, *Archimedes*, pp. 67–68.

But DA is equal to DH;
therefore DH is greater than DG, the less than the greater: which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference."

Now, here is the first part of *Conica* I.32, covering the case of the parabola:

Let there be a section of a cone, first the so-called parabola whose diameter is the straight line AB [see fig. 2], and from A let the straight line AC be drawn parallel to an ordinate.

Now it has been shown that it falls outside the section (I.17).

Then I say that also another straight line will not fall into the space between the straight line AC and the section.

For if possible, let it fall in, as the straight line AD, and let some point D be taken on it at random, and let the straight line DE be dropped ordinatewise, and let the straight line AF be the parameter of the ordinates. And since

$$\text{sq.DE:sq.EA} > \text{sq.GE:sq.EA} \ldots,$$

and

$$\text{sq.GE} = \text{rect.FA,AE} \quad (\text{I.11}),$$

therefore also

$$\text{sq.DE:sq.EA} > \text{rect.FA,AE:sq.EA},$$

or

$$[\text{sq.DE:sq.EA}] > \text{FA:EA}.$$

Let it be contrived then that

$$\text{sq.DE:sq.EA::FA:HA}, \ldots$$

and through the point H let the straight line HLK be drawn parallel to ED.

Since then

$$\text{sq.DE:sqEA::FA:AH::rect.FA,AH:sq.AH}$$

and

$$\text{sq.DE:sq.EA::sq.KH:sq.HA} \ldots,$$

and

$$\text{sq.HL} = \text{rect.FA,AH} \ldots,$$

therefore also

$$\text{sq.KH:sq.HA::sq.LH:sq.HA}.$$

Therefore

$$KH = HL;$$

and this is absurd. Therefore another straight line will not fall into the space between the straight line AC and the section.

Later in I.35,36, where Apollonius sets out the main property of the tangent to a parabola (I.35) and to a hyperbola and ellipse (I.36), one has again the sequence in which first we are given the property of a tangent, that is, of a line touching the curve, and then, told that "no straight line will fall into the space between the tangent and the section." So, although not defined by him, Apollonius's treatment of the tangent to a conic section is such that it would appear completely natural and expected to one familiar with Euclid's treatment of the tangent to a circle in the *Elements*.[13]

Unlike the tangent, however, there is nothing in the *Elements* that one can link to the asymptote of a hyperbola.[14] The only other curve having an asymptote with which Apollonius and his contemporaries might have been acquainted was Nicomedes' conchoid.[15] Proclus, who takes "asymptotes" literally, that is, as any non-intersecting lines, expresses wonders at the type of asymptote connected with the hyperbola and conchoid. He says, ". . . of the asymptotes that are in the same plane, some are always equidistant from one another, others are constantly diminishing the distance between themselves and their straight lines, like the hyperbola and the conchoid. Although the distance between these lines constantly decreases, they remain asymptotes and, though converging upon one another, never converge completely. This is one of the most paradoxical theorems in geometry (ὅ καὶ παραδοξότατον ἐστιν ἐν γεωμετρίᾳ θεώρημα . . .), proving as it does that some lines exhibit a nonconvergent convergence."[16] As Taliaferro notes,[17] Apollonius also refers, here and there, to *any* line not meeting the hyperbola as an asymptote, for example in the

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[13] This point in general will be further developed below, when we discuss the analogy between conic sections and the circle.

[14] Except, perhaps, parallel lines which Proclus includes in the class of non-intersecting lines (αἱ ἀσύμπτωτοι εὐθεῖαι) (cf. *In Eucl.*, pp. 175ff.).

[15] Pappus, *Coll.*IV.26–27. We described the construction, as it appears in Book IV of Pappus's *Collection*, in our chapter on Book V of the *Conica*. In that construction, the line AB, the "rule" (κανών) as Pappus calls it, is the asymptote.

[16] Proclus, *In Eucl.* p. 177).

[17] Taliaferro, *On Conic Sections Books I–III*, p. 683.

porism to II.14, which says that "... from this [*proposition* II.14] it is evident that the straight lines AB and AC [the asymptotes in the specialized sense] [see fig. 3] are nearer than all the asymptotes to the section, and the angle contained by BA,AC is clearly less than that contained by other asymptotes to the section." One should note, however, that the porism cited here follows precisely the proposition demonstrating the "paradoxical" fact that "The asymptotes and the section, if produced indefinitely, draw nearer to each other and they reach a distance less than any given distance,"[18] emphasizing that, like the tangent, which also has a simple literal sense of a line "touching" a curve, the asymptotes in Apollonius's usage are first of all the particular lines referred to in the statement of II.14, rather than lines distinguished only in their "not meeting" the hyperbola. Indeed, the very fact that the porism is stated in connection with II.14 suggests that with II.14 we have reached some final understanding of *what* asymptotes are.[19]

The propositions between II.1 and II.14 concerning the asymptotes, II.1–4,8–14, include some that are quite striking and important for the remainder of the book, for example, II.12, which is the basis for almost every construction involving hyperbolas. However, the propositions among those that seem most closely related to the understanding of asymptotes are, in addition to the porism to II.14, II.1,2, 13 and, of course, II.14. These follow a pattern that is parallel to the development of the tangent, in the sense that after presenting the determining property of the asymptote (II.1), Apollonius refines the notion of an asymptote by showing, essentially, that no other line will fall in the space between the asymptote and the hyperbola (II.2,13). Let us look a little more closely at the development of the asymptote in the sequence II.1,2,13,14. Proposition II.1 states that "If a straight line touch an hyperbola at its vertex, and from it on both sides of the diameter a straight line is cut off equal to the straight line equal in square to the fourth of the figure, then the

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[18] *Conica*, II.14.

[19] Strictly speaking porisms are as Proclus describes them to be, namely, "... a kind of incidental gain resulting from the scientific demonstration ..." (*In Eucl.*, p. 212). In the *Conica*, porisms are rare, but when they do occur, however, they often mark crucial junctures in the text, for example, the porism following I.7 stating that the diameter of a section will not always be an axis (which is one of the first important new insights into conic sections that come directly from Apollonius's way of defining and sectioning the cone).

straight lines drawn from the center of the section to the ends thus taken on the tangent will not meet the section." This gives the *symptoma* of the asymptote, as it were, but in this enunciation the word asymptote in its nominal sense does not yet appear. Only at the end of the proof Apollonius says that "therefore the straight lines CD and CE [where C is the center of the hyperbola and the squares on DB and BE are equal to the fourth part of the figure (see fig. 4)] are asymptotes to the section." One still does not know, however, whether CD and CE are asymptotes in more than the literal sense of lines simply not touching the hyperbola. This is addressed in II.2, which states that "With the same things it is to be shown that a straight line cutting the angle contained by the straight lines DC and CE is not another asymptote." But does this proposition make DC and CE *more properly asymptotes* than other lines through C that do not touch the hyperbola? The answer is not clear until the proof of II.3 where one sees just how II.1 and II.2 work together, in Apollonius's mind, to allow one to say that a pair of lines are *the* asymptotes. II.3 is the converse to II.1 and states that "If a straight line touches an hyperbola, it will meet both of the asymptotes and it will be bisected at the point of contact, and the square on each of its segments will be equal to the fourth of the figure resulting on the diameter drawn through the point of contact." Here is its proof:

> Let there be the hyperbola ABC [see fig. 5], and its center E, and asymptotes FE and EG, and let some straight line HK touch it at B.
>
> I say that the straight line HK produced will meet the straight lines FE and EG.
>
> For if possible, let it not meet them, and let EB be joined and produced, and let ED be made equal to EB; therefore the straight line BD is a diameter. Then let the squares on HB and BK each be made equal to the fourth of the figure on BD, and let EH and EK be joined. *Therefore they are asymptotes (II.1); and this is absurd (II.2); for FE and EG are supposed asymptotes* [emphasis added]. Therefore KH produced will meet the asymptotes EF and EG at F and G.
>
> I say then also that the squares on BF and BG will each be equal to the fourth of the figure on BD.
>
> For let it not be, but if possible, let the squares on BH and BK each be equal to the fourth of the figure. *Therefore HE and EK are asymptotes (II.1); and this is absurd (II.2)* [emphasis added]. Therefore the squares on FB and BG will each be equal to the fourth of the figure on BD.

The proof of II.3 would obviously collapse if the asymptotes FE and EG could be *any* lines through E *not touching* the hyperbola. Yet when

Apollonius says that FE and EG are asymptotes with no further qualification, he is not concerned that he will be misunderstood. This is because II.1 and II.2 have made what *he* means by asymptotes sufficiently clear, even though he never says outright that he "*calls* such lines and *only* such lines asymptotes." Accordingly, one can read the critical parts of the proof of II.3 where Apollonius reasons, "*therefore* HE and EK *are asymptotes (II.1); and this is absurd (II.2); for FE and EG are supposed asymptotes*," as saying, in effect, that "EH and EK cannot be asymptotes, for supposing that FE and EG are asymptotes means both that these lines do not touch the hyperbola and *also* that no other line can be put between them and the hyperbola."

Propositions II.2 and II.3 are used later in the *Conica*. Proposition II.2, for example, is used at the beginning of V.43. Propositions II.13 and II.14, however, are not used again. Their only function seems to be to put into clearer focus the idea that no other line can be put between the asymptotes and the hyperbola. II.2, after all, only shows this in a specific circumstance, namely, when one endpoint of the line is at the center of the section, or, in general, when the line forms an angle with the asymptote.[20] Proposition II.13 then uses II.2[21] to show that a line not forming an angle with the asymptote, that is, a line parallel to the asymptote, also cannot be interposed between the asymptote and hyperbola. The second part of II.14 sums up II.2 and II.13 by saying that the distance between the asymptote and the hyperbola becomes smaller than any given distance.

The first part of II.14, however, adds something new, namely, that "The asymptotes and the section, if produced indefinitely, draw nearer to each other . . ." The proof of this part is simple: with the asymptotes of the hyperbola being AB and AC [see fig. 3], ". . . let

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[20] That the vertex of the angle is the *center* does not restrict the claim. For suppose one were to say that some line forming an angle with the asymptote could be interposed between the asymptote and the hyperbola, provided that the vertex of the angle was some point *other* than the center. Then if the vertex were "farther" from the hyperbola than the center (*i.e.*, the perpendicular from the point to the axis intersects the axis at a point p with the center between p and the vertex of the section with respect to the axis), a line could be drawn through the center *parallel* to the line forming the angle with the asymptote, and this clearly would not cut the curve. Furthermore, if the vertex were "closer" to the hyperbola, then a line could be drawn from the center *intersecting* this line (by Euclid's 5th postulate in the *Elements*) between the asymptote and the hyperbola, and this too would not cut the curve since to do so would mean cutting the curve twice.

[21] Taliaferro has "I.2" in the relevant place; this is obviously a misprint. (Corrected in the Green Lion reprint).

EHF and CGD be drawn parallel to the tangent,[22] and let AH be joined and produced to X. Since then

rect.CG,GD = rect.FH,HE . . .,

therefore

DG:FH::HE:CG.

But

DG > FH . . .;

therefore also

HE > CG.

Then likewise we could show that the succeeding straight lines are less."

Taliaferro has I.8 and I.26 justifying the step DG > FH. The former is not really appropriate here since it only says that in a parabolic and hyperbolic section if ". . . the surface of the cone and the cutting plane are produced indefinitely, then the section will also increase indefinitely, and some straight line drawn from the section of the cone parallel to the straight line in the base of the cone will cut off from the diameter on the side of the vertex a straight line equal to any given straight line." Proposition I.8, therefore, only says that the hyperbola increases indefinitely in the direction of the diameter; it does not say that lines drawn parallel to that in the base of the cone increase indefinitely. Proposition I.26 *can* be used to justify the step as follows: suppose QHG is a hyperbola (see fig. 6) for which AFD is an asymptote, FH and DG are parallel, and BAWIJ is the diameter with respect to which FH and DG are the ordinate directions. Suppose, also, that DG were less than FH. Since DJ is greater than FI, JG must be less than IH *a fortiori.* Draw GP through G parallel to the diameter AIJ. Then, since JG < IH, GP must intersect IH at a point P *between* I and H. But then GP must intersect the hyperbola at a point Q in the segment WQH,[23] contradicting

[22] There seems to be a lacuna here since no tangent has been given or drawn. It is possible that Apollonius intends the tangent at the vertex by means of which the asymptotes are first drawn in II.1. Knowing which tangent is meant, however, does not affect the proof; the application of II.10, in particular, only requires that EHF and CGD be parallel to one another.

[23] Of course, we are assuming here a principle of continuity, something like what is assumed in *Elem.*I.1.

I.26 which says that "If in a parabola or hyperbola a straight line is drawn parallel to the diameter of the section, it will meet the section in one point only." Of course, the easiest proof is by means of I.21,[24] for by that proposition we have (fig. 6), sq.IH:sq.JG::rect.WI,IB: rect.WJ,JB. But WI < WJ and IB <JB, therefore, rect.WI,IB < rect. WJ,JB, from which we have sq.IH < sq.JG and IH <JG. The point is that the statement DG > FH is not immediate, at least it is *no more* immediate than the very thing Apollonius is trying to prove. Perhaps, someone might argue, Apollonius so much more carefully proves HE > CG in II.14 (fig. 3) because he wants another impressive application of II.10, besides that in II.11; indeed, II.10 is essential in II.14, for, *without* II.10, it might be possible that DG > FH while HE < CG, for example, if the chord HG met the asymptote AC between A and E (see fig. 7), that is, just because the hyperbola recedes continuously from one asymptote does not mean it approaches the other. But given the weight of II.14 in determining our understanding of what asymptotes are, as we argued above, it would be strange if the first fact proven in II.14, that the asymptotes and the hyperbola draw nearer to each other, were important to Apollonius only because it is an opportunity to show the use of II.10.

Considered in light of the tangent, however, one obtains an additional reason for the importance of the first part of II.14. One of the peculiarities of the tangent to the circle, as set out in *Elem.*III.16, is that it makes an angle with the circumference less than any rectilinear angle. The proof of this in *Elem.*III.16 (and that the angle made by the diameter and the semicircle is greater than any acute angle) goes hand in hand with the proof that no line can be interposed between the circle and the tangent. For, having said that ". . . another straight line cannot be interposed into the space between the straight line and the circumference," Euclid continues:

> I say further that the angle of the semicircle contained by the straight line BA [the diameter] and the circumference CHA [see fig. 1] is greater than any acute rectilineal angle, and the remaining angle contained by the circumference CHA and the straight line AE [the tangent] is less than any acute rectilineal angle.

[24] In their edition of the Taliaferro translation, Green Lion has put *Elem.*VI.4, and CN 5 in place of Taliaferro's original citation of I.8, 26. The argument suggested, then (see fig. 3), is this: DX > FH by Elem.VI.4 and DG > DX by CN 5, therefore, DG > FH. It is an acceptable argument.

> For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference CHA, and any rectilineal angle less than the angle contained by the circumference CHA and the straight line AE, then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.
>
> But such a straight line cannot be interposed; therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.

Now, the idea of an angle itself was greatly discussed in antiquity.[25] Proclus stresses, for example, that there was no clear agreement as to the proper category for angle, whether it should be considered a relation, a quality, or a quantity.[26] Proclus himself thinks that an angle "... exists as a combination of all these categories, and this is why it presents a difficulty to those who are inclined to make it any one of them."[27] Part of the problem is that a plane angle need not be contained by two straight lines, that is, be rectilineal, but can be contained as well by two curved lines, or a curved and straight line (such as in the case of the circle and one of its tangents, which results in the so-called "horn angle"), or even a curved line with itself, as in the cissoid and hippopede. According to Proclus, Apollonius had his own thoughts as to what an angle is, saying that it is a "... contracting (συναγωγή) of the surface at a point under a broken line ...,"[28] as opposed to Euclid's inclination (κλίσις) of one line to another.[29] An asymptote and the curve with which it is associated do not contain an angle since they have no point in common, and having such a point in common, despite differences in the definitions of an angle, is evidently shared by all the definitions. However, the

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[25] See Proclus, *In Eucl.*, pp. 121–126, Heath, *Euclid*, vol. 1, pp. 176–179.

[26] Proclus, *Ibid.*, p. 121.

[27] *Ibid.*, p. 123.

[28] *Ibid.*

[29] Euclid's definition (*Elem.*I, def.8) in its entirety is as follows: "A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line."

fact that the asymptote and the hyperbola draw nearer to each other gives it some similarity to an angle, whether the latter be thought of as the contraction of a surface or of the inclination of two lines to one another. Indeed, what Proclus finds paradoxical about the asymptote is precisely that it grows ever closer to the hyperbola and yet does *not* join with it, that the existence of asymptotes shows that there are "... some lines [that] exhibit a nonconvergent convergence (σύνευσιν τινῶν γραμμῶν ἀσύνευστον)."[30] In other words, what makes the asymptote paradoxical is that it involves something so much *like* an angle, in particular like the "horn angle" between a tangent and the circumference of a circle, without actually *being* an angle. With the angle-like character of the asymptote and the hyperbola in mind, the first part of II.14 can be seen to correspond, in this respect, to Euclid's tangent making an angle with the circumference smaller than any rectilinear angle.[31] This strange creature, the asymptote, if not, therefore, made more intelligible by its connection with the tangent, is transformed at least into a notion that can be related to one connected with the very basic shape, the circle.[32]

Circles and Conic Sections

In the introduction to this chapter, we termed the analogy between the circle and the conic sections an external analogy. This is a debatable characterization. For, assuming that there is, in fact, an analogy between the circle and conic sections—and we shall show that

[30] Proclus, *In Eucl.*, p. 177.

[31] The analogy between the tangent and the asymptote is, in this respect, a "negative" analogy, to use Keynes's term (*cf.* Lloyd, *op. cit.*, p. 175), but a negative analogy is still an analogy, and a strong one at that. We should add that this analogy between the tangent and asymptote makes Knorr's remark that "It is striking that neither geometer [Apollonius nor Nicomedes] attempts to frame the treatment of asymptotes in the manner of Eudoxus and Archimedes, for instance, by means of a limiting procedure based on successive bisection" (*Ancient Tradition of Geometric Problems*, p. 224), somewhat less striking.

[32] At this point, we want to emphasize that our categorizing the connection between the tangent and the asymptote as an *analogy* is a reaffirmation that the tangent and asymptote are, indeed, *different* entities. The obverse claim, namely, that the asymptote *is a kind* of tangent "at infinity" (cf. Heath, *History*, II,155), is utterly anachronistic and goes in a completely different direction from that of the discussion above.

there is and that it is intentional—must it not, if anything, be an *internal* analogy? Is it not true that the circle is itself a conic section? Indeed, if it has properties in common with the other conic sections, is it not merely because it is generated with them from the cone? Yes, the circle is a conic section; Apollonius proves it so. Yet, the circle is also distinguished from the other conic sections in that it is known in a manner completely free of any consideration of the cone, or of any other solid; it is a plane figure. Thus, for example, problems involving the circle are considered "plane," and not "solid."[33] For this reason, we have chosen to view the circle as an object outside the main thrust of the *Conica*, and, therefore, an object that can serve as the basis of an *external* analogy. In this last section, we shall show first that Apollonius himself separates the circle from the other conic sections, and, second, that in the *Conica* the circle is ever in the background as a kind of model for the other sections.

The first aspect of the text in which one detects a separation between the circle, on the one hand, and the parabola, hyperbola, ellipse, and opposite sections, on the other, is in the placement of the propositions that introduce the respective sections. The propositions which introduce the parabola, hyperbola, ellipse, and opposite sections, propositions I.11–14, form a distinct group and mark the beginning of the second part of Book I. Proposition I.4 introducing the circle as a conic section is located in the middle of the part of Book I, which we called earlier the "mini-treatise" on cones and sectioning. So, one might say that the separation here is a literal one. By itself, however, the observation is nugatory, since I.4 is required in the proofs of propositions I.5, I.8, and I.9. Furthermore, proposition I.8, at least, clearly belongs among the propositions outlining the general properties of sections, so that I.4 really has to stay where it is. That said, one *could* conceive of a *Conica* where proposition I.5 were used to define the circle-section and proposition I.9 were absorbed into I.13. As for the latter, the enunciation of I.13, as it

[33] In this context, we should mention the well known and much discussed (for example, Zeuthen, *Die Lehre*, p. 286; Heath, *Apollonius*, pp. cxxviii–cxxix, Knorr, *Ancient Tradition of Geometric Problems*, pp. 319–321) comment by Pappus in Book IV of the *Collection* (pp. 270–272) that problems solvable by straight lines and circles, that is, "plane" problems, should not be solved by "conics or other [curved] lines," and that Apollonius was guilty of this in regarding the problem of the parabola in the fifth book of the *Conica* (it is not completely agreed as to *which* problem this is).

stands, begins with the words "If a cone is cut by a plane through its axis, and is also cut by another plane on the one hand meeting both sides of the axial triangle, and on the other extended neither parallel to the base nor subcontrarwise . . ." whereas I.9 begins with the words "If a cone is cut by a plane meeting both sides of the axial triangle, and neither parallel to the base nor situated subcontrariwise . . ."; so, with the first conditions of the propositions, thus, being the same, the absorption of I.9 into I.13 could be carried out quite naturally. As for the former, a proposition defining a circle-section could precede proposition I.13 and be stated as follows: "If a cone is cut by a plane through the axis at right angles to the base, and also cut by another plane on the one hand at right angles to the axial triangle, and on the other cutting off on the side of the vertex a triangle similar to the axial triangle, then the section will be a circle, and if the triangle lies subcontrariwise then the circular section will be called subcontrary." The proof would then proceed by considering the two cases, namely, when the bases of the similar triangles are parallel and when they are not. The first case would be disposed of by citing I.4, while the second case would have the proof of I.5.

The reason Apollonius does *not* follow a scheme like this, however, has already been explained in our chapter on the elements of conic sections: propositions I.1–10 together provide the fundamental properties of cones and sections of cones, and, within this framework, I.4, I.5, I.8, and I.9 have the function of defining the different modes of sectioning a cone and the very basic properties of the sections produced. What we should like to stress here, in addition, is that the circle produced in proposition I.4 is used in each of these proposition in a critical way. This is particularly pronounced in I.9, where, having shown in I.8 that there are sections that extend indefinitely, it is shown that there are sections that do not extend indefinitely[34] but which are not circles. Later, in proposition I.13, we learn that this section is to be called the ellipse and we learn its fundamental property. Proposition I.9, hence, gives us a first sketchy introduction to the ellipse, and what we are told about it is that it

[34] This is not stated explicitly, but it is clear, since the cutting plane cuts the two sides of the axial triangle joined at the vertex of the cone, and, therefore, there can be, it seems, an entire section contained between the vertex and the base in addition to the circle.

is not a circle. The comparison with the circle, whose important properties are well understood and well known, is, therefore, the basis of our first understanding of the other conic sections, as yet unfamiliar and unknown. So, the placing of these propositions dealing exclusively with the circle *apart* from the propositions defining the other conic sections is, beyond any deductive necessity, representative of the circle's different *status* within the study of conic sections.

In the rhetoric and substance of individual propositions, the circle is also separated from the other sections. Thus, to take a random example, III.1 states "If straight lines, touching *a section of a cone or circumference of a circle* [emphasis added], meet, and diameters are drawn through the points of contact meeting the tangents, the resulting vertically related triangles will be equal." Sometimes, the circle appears in a list together with the ellipse and hyperbola, as if it were just another section, but, of course, such lists do *not* contain the parabola; if the proposition referred to the hyperbola, ellipse, circle, *and parabola*, Apollonius would, no doubt, revert to the former formulation ". . . a section or circumference of a circle." Conversely, there are propositions that refer to a section of a cone *in general*, but whose demonstration neither proves nor even mentions the case of the circle. This occurs, for example, in the problem II.49, which states, "Given a section of a cone and a point not within the section, to draw from the point a straight line touching the section in one point," but which is proven only for the parabola, hyperbola, and ellipse. Now, it could be argued that Apollonius does not need to mention the case of the circle here since that is proven in *Elem.*III.17. But this raises a question as much as it answers one. For the demonstration of II.49 applied to the circle would not be the same as that of *Elem.*III.17, as the latter uses the perpendicularity of the diameter and tangent proven in *Elem.*III.16, while *Conica* II.49 depends on *Conica* I.36, which proves for the hyperbola, ellipse, *and circumference of a circle*, that if AB is a diameter (see fig. 8), CD is a tangent at C, and CE is dropped ordinatewise to AB, then BE:EA::BD:DA. Apollonius *could* have shown that, just as I.36 solves the problem of finding a tangent from a point outside a hyperbola and ellipse, it also solves the problem for the circle as well, providing, in this way, an *alternative* method for solving the problem set out in *Elem.*III.17. In any case, the argument that the circle is not mentioned in II.49 only because the case of the circle is proven in *Elem.*III.17, cannot be made consistently: the argument does not work, for example, for

proposition II.26, which states, "If in an ellipse or circumference of a circle two straight lines not through the center cut each other, then they do not bisect each other," since this, in the case of the circle, is precisely the statement of *Elem.*III.4. Nor does the argument work for (*Conica*) III.17, where Apollonius proves that "If two straight lines touching a section of a cone *or circumference of a circle* [emphasis added] meet, and two points are taken at random on the section, and from them in the section are drawn parallel to the tangents straight lines cutting each other and the line of the section, then as the squares on the tangents are to each other, so will the rectangles contained by the straight lines taken similarly." In the case of the circle (a diagram of which, by the way, is included together with those of the parabola, hyperbola, and ellipse), the tangents will always be equal, hence, the proposition in this case says that the rectangles contained by the segments of the intersecting straight lines taken similarly will be equal. This, however, is proved in *Elem.*III.35. If the argument above were true, then in both these examples, II.26 and III.17, bringing up the case of the circle would be unnecessary.

In fact, in the propositions where the circle is mentioned explicitly, even when one does *not* find a particular proposition in Euclid covering that case, one can almost always construct a simple proof for the circle that does not go beyond the *Elements*, so that, for the circle, the machinery developed in the *Conica* is almost never essential. Consider, for example, I.45. It states, "If a straight line touching an hyperbola or ellipse or circumference of a circle meets the second diameter, and from the point of contact some straight line is dropped to the same diameter parallel to the other diameter, and through the point of contact and the center a straight line is produced, and, some point being taken at random on the section, two straight lines are drawn to the second diameter one of which is parallel to the tangent and the other parallel to the dropped straight line, the triangle resulting from them is greater, in the case of the hyperbola, than the triangle the dropped straight line cuts off from the center, by the triangle whose base is the tangent and vertex is the center of the section, and, in the case of the ellipse and circle, together with the triangle cut off will be equal to the triangle whose base is the tangent and whose vertex is the center of the section." The demonstration for the ellipse and hyperbola depends crucially on I.39, but this is not so for the circle. For suppose ABC is a circle with center H (see fig. 9). Furthermore, suppose AH is a diameter,

HD is a "second diameter" (and, therefore, perpendicular to AH), and CML touches the circle at C. Let B be taken at random on the circle, BE is drawn parallel to CL, BF drawn parallel to AH, and CH joined. Thus, we need to prove that,

trgl.BEF + trgl.FGH = trgl.CLH.

Since ABC is a circle it follows that CH is perpendicular to CL. Also, by construction, DH is perpendicular to AH (and, accordingly, BF to EF). Therefore, triangles CLH, FEB, and FGH are similar. Therefore, trgl.BEF:trgl.GFH::sq.BF:sq.FH, or, *componendo*, trgl. BEF + trgl.GFH:trgl.GFH::sq.BF + sq.FH:sq.FH. But, sq.BF + sq.FH = sq.BH = sq.CH, therefore, trgl.BEF + trgl.GFH: trgl.GFH:: sq.CH:sq.FH. On the other hand, since triangle CLH is similar to FGH, we have, trgl.CLH:trgl.FGH::sq.CH:sq.FH. Therefore, trgl.BEF + trgl.GFH:trgl.GFH:: trgl.CLH:trgl.FGH, and, therefore, trgl.BEF + trgl.GFH = trgl.CLH.

In light of examples, such as the above, it becomes hard to maintain that when Apollonius refers to the circumference of a circle in a proposition he intends for us actually to apply the proof intended for the conic sections proper specifically to the circle. Of course, this is not to say that the proofs are not valid for the circle, but actually using them to prove something positive about the circle is all too often like killing a flea with a sledge hammer, not only because of the length and complexity of the proofs (a length and complexity that is unavoidable in the cases of the hyperbola and ellipse), but also because the very terms used in the proof are forced in the case of the circle. In particular, when Apollonius refers to the *latus rectum* of the conic sections, or the "figure," or the "second diameter" of a conic section, one must remember that these things are never defined explicitly for the circle.[35] So, when in I.21 Apollonius says

[35] One might be tempted to argue that Apollonius does not have to define these things for the circle because they are defined for the ellipse, and the circle is a kind of ellipse. This argument is unacceptable since it is fairly clear that *Apollonius did not see the circle as a kind of ellipse.* Had he thought otherwise, then there would have been no need for him to refer to the circle after he has already referred to the ellipse. Indeed, in I.13 where Apollonius defines the ellipse he specifically rules out the cutting plane cutting the cone parallel to the base or subcontrariwise, and by this he rules out all sections producing a circle. *Our* tendency to see the circle as a special case of an ellipse, in fact, arises out of a subtle *algebraic* bias in our

"If in an hyperbola or ellipse *or in the circumference of a circle* [emphasis added] straight lines are dropped ordinatewise to the diameter, the squares on them will be to the areas contained by the straight line cut off by them beginning from the ends of the transverse side of the figure, *as the upright side of the figure is to the transverse* [emphasis added], and to each other as the areas contained by the straight lines cut off (abscissas), as we have said," one can fit the proof to the circle by taking the upright side of the figure equal to transverse side of the figure, that is, to the diameter of the circle, as we mentioned in the chapter on maximum and minimum lines,[36] but, with that, what we learn from I.21 concerning the circle is only that the squares on the ordinates are equal to the rectangles contained by the segments cut off from the diameter, a fact that Apollonius has assumed almost from the beginning of the book! So, in what way are we to understand these references to the circumference of a circle? For the most part, we think Apollonius refers to the circle not because he wants to prove something new about the circle, but because he wants to make an implied comparison, to the effect that a proposition is true regarding the hyperbola and ellipse *as it is true regarding the circle.*

That this comparison with the circle is ongoing throughout the *Conica* is supported by the numerous parallels, which we cited here and elsewhere in the present work, between propositions in the *Conica* and in Book III of the *Elements.* Indeed, we noted, in connection with Apollonius's efforts in Book V and Euclid's *Elem.*III.7,8, that such parallels were also noticed by Eutocius in his commentary on the *Conica.* Furthermore, we observed in our chapter on the elements of conic sections that the terms "diameter," "center," and "radius," defined for the conic sections in Book I are clearly drawn from the

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view of conic sections. For, in the algebraic view, the *latus rectum* is taken as the *parameter* p in the *equation* for the ellipse $y^2 = \frac{p}{d} x (d - x)$. This means that, as long as p is positive, p can vary without changing the equation of an ellipse—one simply obtains different ellipses, *i.e. different equations of the same form.* In particular, when p is equal to the positive number d (equal to the length of the transverse diameter), one still has an ellipse. But when $p = d$, the equation becomes the equation of a circle. So since the equation of the circle is, therefore, a special case of the equation of the ellipse (that is, it is arrived at through a particular choice of the *parameter*), the circle *itself,* in the algebraic view, is a special case of the ellipse.

[36] Chapter IV, note 59.

corresponding terms defined by Euclid for the circle. In this context, the last is particularly revealing, for Apollonius hardly ever uses the term "radius," which he defines together with "center" of a conic section after I.16.[37] Knowing that lines are radii of a circle can be quite significant since such lines are then equal; this is the main source of a circle's usefulness in constructions, as Apollonius well knows.[38] By contrast, knowing that lines are radii of a conic section does not in itself appear to have the same force. Apollonius uses the term in two propositions, II.22,23. In the first of these, Apollonius writes, "If in conjugate opposite sections *a radius* [emphasis added] is drawn to any one of the sections, and a parallel is drawn to it meeting one of the adjacent sections and meeting the asymptotes, then the rectangle contained by the segments produced between the section and the asymptotes on the straight line drawn is equal to the square on *the radius* [emphasis added]." What does Apollonius gain by using the same word "radius" here instead of simply speaking, as he does in II.20, of a line drawn through the center? Apparently, not very much, for he never uses II.22,23 (except for II.22 in II.23), whereas he does use II.20 more than once afterwards.[39] Given this marginal use of the term, the plausible alternative as to why Apollonius defines the "radius" is only that he wants to remind the reader of the connection between the conic section and the circle. Let us look now at one more example in which Apollonius's treatment of an aspect of conics is related to that of the circle.

Book IV of the *Conica*, unlike the other elementary books and somewhat more like the later books, focuses sharply on a single theme and its propositions are not used afterwards in the *Conica*. The theme of the fourth book is, of course, the possible ways in which conic sections can meet; it can be framed, therefore, as an elaborate book-long development of two propositions in Euclid's *Elements*, III.10,13, propositions which, incidentally, are also never

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[37] There is no single specific word for "radius" in Greek mathematical terminology, only the substantive phrase ἡ ἐκ τοῦ κέντρου, "that from the center." Mugler points out (p. 164) that Apollonius's definition is remarkable since in his description of the radius he uses the preposition ἀπό, conforming to contemporary usage, while for the term itself he reverts to the earlier usage of ἐκ. This would very likely have the effect of calling attention to the more traditional use of the phrase in connection with the circle.

[38] Cf. II.47,48.

[39] For instance, in II.45 and VII.31.

used by Euclid beyond Book III.[40] *Elem.*III.10 states that "A circle does not cut a circle at more points than two," while *Elem.*III.13 states that "A circle does not touch a circle at more points than one, whether it touch it internally or externally." The analogues to *Elem.*III.10 in the *Conica* are IV.25,38,55. Proposition IV.25, for instance, states that "A section of a cone does not cut a section of a cone or the circumference of a circle in more points than four," while IV.38 says that "A section of a cone or circumference of a circle will not intersect the opposite sections in more points than four," and IV.55, that "Opposite sections do not cut opposite sections in more points than four."[41] The elaboration of *Elem.*III.13 in the *Conica* is naturally much more involved than that of *Elem.*III.10 because of the great diversity of ways in which conic sections can touch and intersect one another. However, we can take IV.26,27 as emblematic of the series of propositions in Book IV related to *Elem.*III.13. Proposition IV.26 states that "If the lines spoken of [*viz.*, sections of a cone or the circumference of a circle] touch one another at one point they will not meet each other at more than two other points," and IV.27, that "If the lines, spoken of before [those in IV.25 and 26], touch one another in two points, they will not meet in any other point."

Eutocius too notes here a connection between the propositions in Euclid and Book IV of the *Conica*,[42] though, interestingly enough, what he notes especially is their use of proofs by contradiction. This is interesting because the contradictions arrived at in, say, *Elem.*III.10 and *Conica* IV.25 are not, on the face of it, the same. In proposition IV.25, Apollonius supposes that two sections meet at *five* points A, B, Γ, Δ, E (see fig. 10). He joins AB and ΓΔ, which, extended,

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[40] In fact, even within Book III, *Elem.*III.10 is only used in *Elem.*III.24 and *Elem.*III.12 is not used at all. This means that these propositions were not likely to be viewed as lemmas, or otherwise ancillary to other results, but as basic facts about circles without which one's understanding of circles, in Euclid's view, would be incomplete.

[41] The reader would do well to recall from our chapter on "Elements" that the opposite sections are problematic for Apollonius—and, as we said in that chapter, this is expressed, among other ways, in Apollonius's separating the opposite sections from the other sections of a cone here in Book IV.

[42] Eutocius does not refer to specific propositions in the *Elements*, but only to propositions proven by Euclid "... about circles' cutting and touching" (περὶ τῶν τομῶν τοῦ κύκλου καὶ τῶν ἐπαφῶν). Heiberg rightly indicates *Elem.*III.10 and 13 as the propositions Eutocius intends.

meet at some point Λ outside the sections (in the case of parabolas and hyperbolas, Apollonius's first case and the only case we shall consider here). AB is then divided at O such that AΛ:ΛB::AO:OB while ΓΔ is then divided at Π such that ΔΛ:ΛΓ::ΔΠ:ΠΓ. Then by IV.9, the line OΠ will, if extended, meet the section[43] at points P and Θ and the lines ΘΛ and ΛP will be tangents to the section. So if EΛ is now joined and intersects PΘ at N, one of the sections at H and the other at M, then we have, by III.37, EΛ:ΛH::EN:NH and, at the same time, EΛ:ΛM::EN:NM, which is impossible. The immediate root of the trouble is that EΛ must pass through the sections in distinct points, H and M. Now, in *Elem.*III.10, Euclid assumes that a circle ABC (see fig. 11) cuts another circle DEF at more than two points, at B, G, F, H. He joins BH and BG, and draws their respective perpendicular bisectors, AC and NO. These line meet at a point P, which must, therefore, be the center of circle ABC as well as DEF. This is impossible because *Elem.*III.5 demonstrates that "If two circles cut one another, they will not have the same centre." So this contradiction seems to be quite different from that in IV.25. However, some similarity appears when one considers the proof of *Elem.*III.5, which is used *only* in *Elem.*III.10, and, therefore, can be safely viewed as a lemma to *Elem.*III.10. As with *Elem.*III.10, the proof of *Elem.*III.5 is also by contradiction. Euclid assumes that circles ABC (see fig. 12) and CDG cut one another at points B and C, but that they have a common center E. Then, with EC joined and some other line EFG drawn across, EC must be equal to EF since E is the center of ABC, EC must also be equal to EG since E is the center of circle CDG. Therefore, EF must be equal to EG, the less to the greater, which is impossible. Here too, then, the immediate root of the problem is that EFG cuts both circles in distinct points, F and G. It may well be then that it was this particular visual situation, also at the crux of the particular contradiction arrived at in IV.25, and a situation indeed recurring throughout Book IV, that Eutocius had in mind when he remarked on the use of contradiction in both Book IV and in the *Elements*; for, surely, the *mere* use of contradiction could not be enough to compare two works. This correspondence between the methods of proof that Apollonius and Euclid

[43] Apollonius does not say *which* of the two sections. However, it is precisely *because* P and Θ can be on one section *or* the other that the contradiction in the end arises.

employ, even though it be only at a certain level, naturally, then, gives further strength to the already unmistakable correspondence between the themes in Book IV and *Elem.*III.10,13.

The examples above as well as others elsewhere in this work show convincingly, we think, that the circle has a powerful presence in the *Conica*. Equally prominent is the specific treatment of circles in Book III of Euclid's *Elements*. Did Apollonius base the *Conica* on this book in the *Elements*? Probably not, or at least not entirely, for there were certainly other advanced geometrical works from which Apollonius could draw material; we know at least that mathematicians such as Conon of Samos, whom Apollonius mentions in the preface to Book IV and who died in Archimedes' lifetime,[44] dealt with such matters as the number of points at which conic sections can meet. Euclid also apparently wrote a book on conics, and, as we know from Apollonius's letter prefacing the first book, Apollonius was acquainted with some of Euclid's work in conics, if not with the (now lost) book itself. Even on the subject of circles, from what can be gathered from Pappus, Euclid's work went beyond what he put into the *Elements*, and it is likely that Apollonius was well aware of these more advanced results. Thus, among Pappus's lemmas to Euclid's *Porisms*, for instance, we find the following proposition which is plainly related to *Conica* III.37: "Let ΑΔ,ΔΓ be tangent to circle ΑΒΓ [see fig. 13], and let ΑΓ be joined, and let an arbitrary (line) ΔΒ be drawn across. That, as ΒΔ is to ΔΕ, so is ΒΖ to ΖΕ."[45] Still, the parallels between the *Conica* and Book III of the *Elements*, as we have seen, *are* quite remarkable,[46] and this leads us to believe that, while Apollonius probably did not *base* the *Conica* on the third Book of the *Elements*, he had that book, or one like it, in mind when he conceived the whole

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[44] Conon's death is mentioned in the letter introducing Archimedes' *Quadrature of a Parabola*. This puts Conon in the generation preceding Apollonius.

[45] *Coll.*VII.154, p. 904 (Hultsch), Jones' translation, p. 284.

[46] Heath sets out some of these examples (in his *Apollonius of Perga*, pp. xcv–xcvi), conveniently providing Euclid's text facing Apollonius's text to allow a truly point by point comparison. He says in the same place, "The closeness with which Apollonius followed the Euclidean tradition is further illustrated by the exact similarity of language between the enunciations of Apollonius' propositions about the conic and the corresponding propositions in Euclid's third Book about circles" (p. xcv). Knorr refers to these correspondences in somewhat stronger terms calling them "conscious imitation" (*Ancient Tradition of Geometric Problems*, p. 313).

of the *Conica*, and, that it, to some extent, guided his thinking about the sort of things his treatise should contain.

Conclusion

What kind of book is a book like the third Book of the *Elements*? First of all, it is a book about the circle, and, it is almost needless to say, the circle held a very prominent place in Greek thought in general. Indeed, admittedly influenced by Plato's *Timaeus*, Proclus calls the circle "The first and simplest and most perfect of the figures . . ."[47] and goes on to say that ". . . we must also contemplate the entire series to which the circle gives rise. Beginning above and ending in the lowest depths of things, it perfects all of them according to their suitableness for participation in it."[48] Beyond the circle's being the best known curvilinear figure whose properties and usefulness any mathematician worth his salt surely knew intimately, we tend to think that the sentiment in Proclus's remarks above held some sway even for Apollonius. Second, Euclid's Book III is a book completely dedicated to a single curved line; it is different, in this respect, from the other books of the *Elements* which are each dedicated, more or less, to a single set of related ideas rather than to a single object. The book brings together various themes concerning the circle, the ways in which circles can meet, greatest and least lines in circles, similarity,[49] area relations,[50] and so on. Euclid's inclusion of such a variety of themes, meant to see what basic things can be said about a given object, is, we think, one explanation why the book tends to be, ". . . from the standpoint of logic, very loosely organized," as Mueller has pointed out.[51] Given the concrete mathematical comparisons between the circle and the conic sections outlined above and in previous chapters, it is plausible that Euclid's

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[47] *In Eucl.* pp. 146–7.

[48] *Ibid.*, p. 148.

[49] Notable, as remarked in our chapter on equality and similarity of conic sections, because it is similarity without an appeal to ratio and proportion.

[50] We have in mind III.35–37. The relation of areas of circles themselves is, of course, considered later in the *Elements*, in Book XII, proposition 2.

[51] Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's* Elements, p. 177.

little treatise on the circle served for Apollonius as a kind of a model treatise on a curved line, a kind of *schematic version* of the (not so little) treatise on conic sections that he set out to write, and it is in this sense that Euclid's book could have guided Apollonius's writing of the *Conica.*

We ought to stress here that when we speak of Apollonius's writing of the *Conica* we mean the writing of the *Conica* as a whole. But are we right truly to consider the book as a whole? While the present chapter has given evidence for affirming this, the wholeness of the *Conica* needs to be addressed further and set out more clearly. To this we must now turn.

Figures

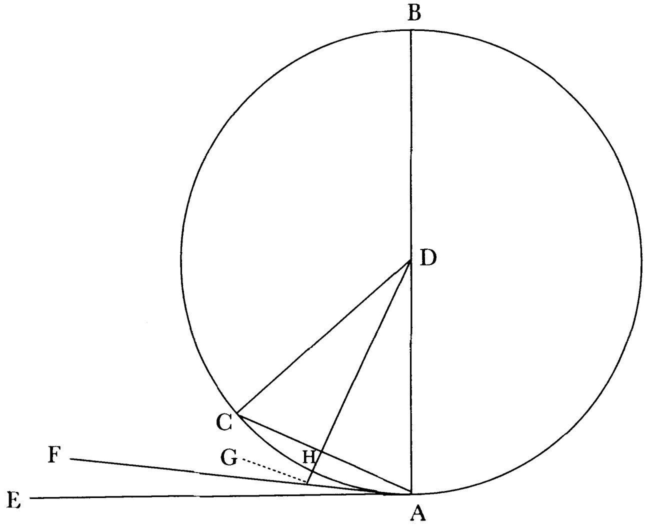

Fig. 1

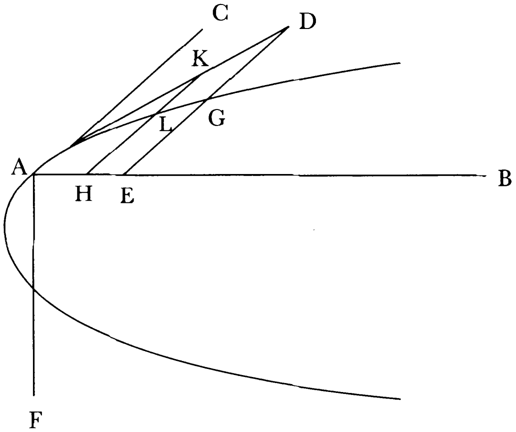

Fig. 2

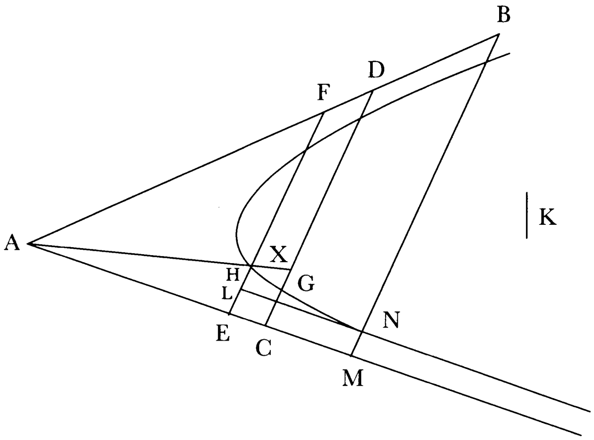

Fig. 3

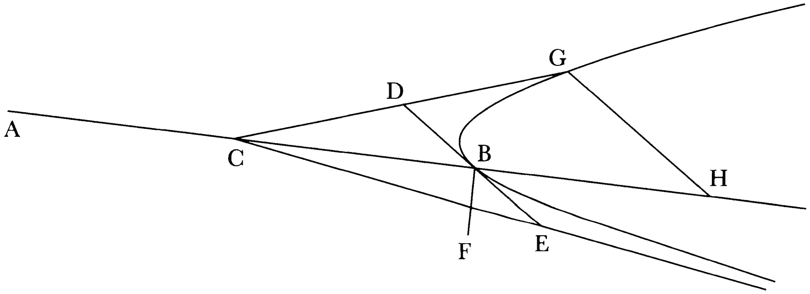

Fig. 4

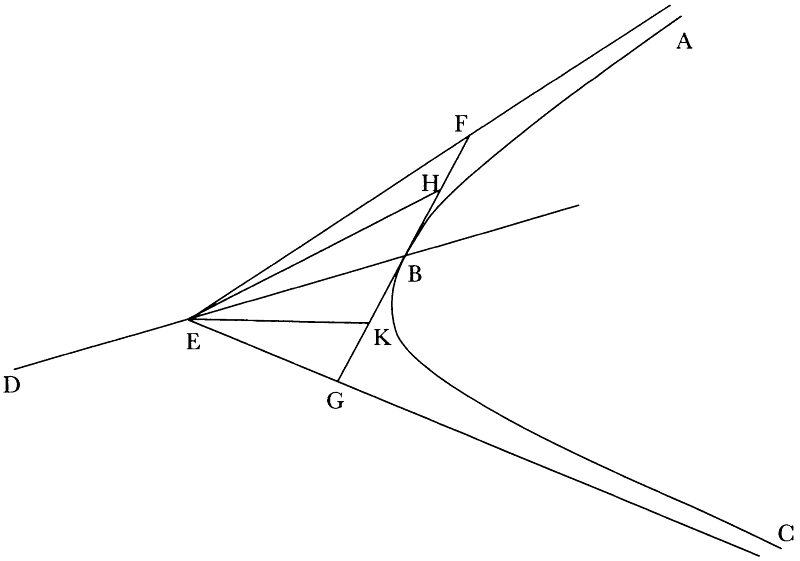

Fig. 5

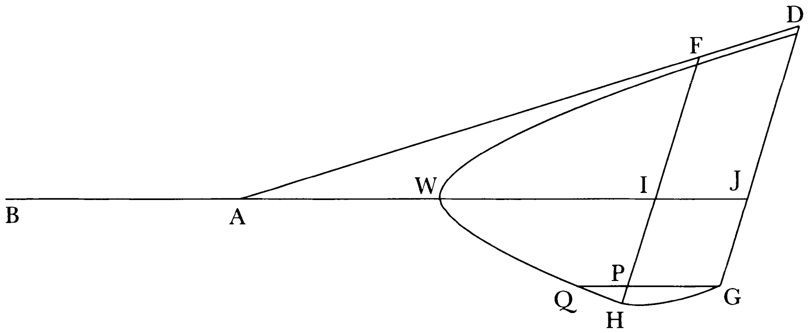

Fig. 6

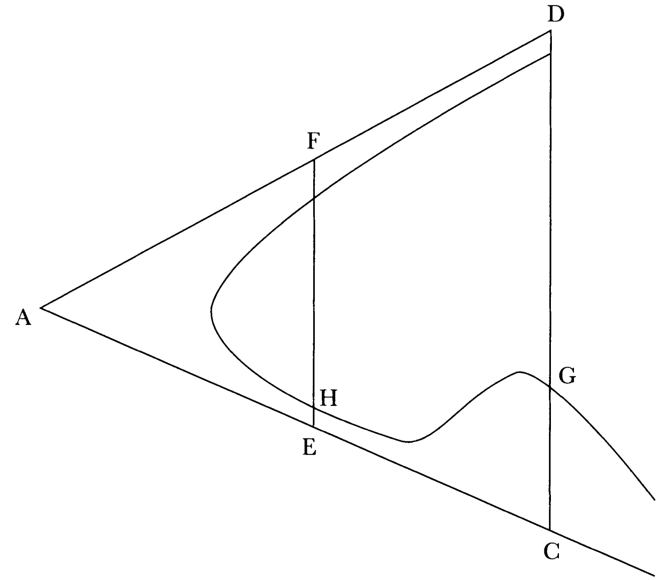

Fig. 7

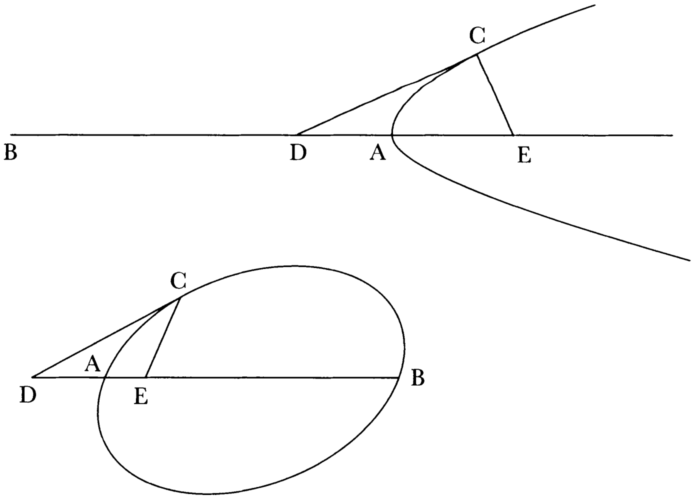

Fig. 8

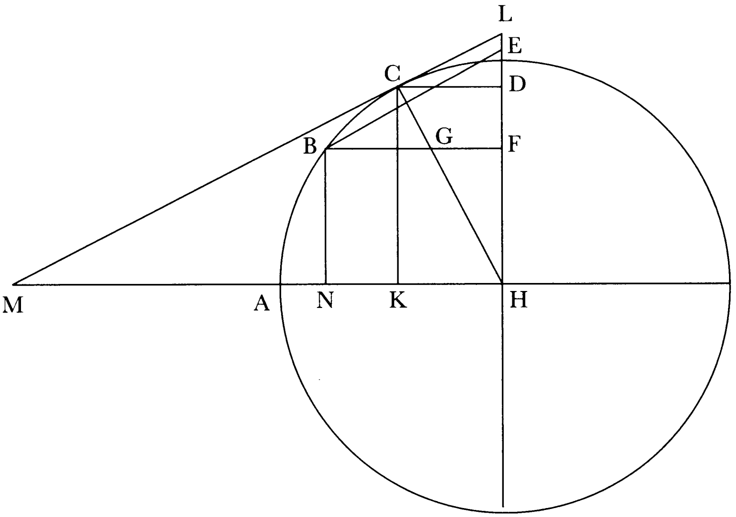

Fig. 9

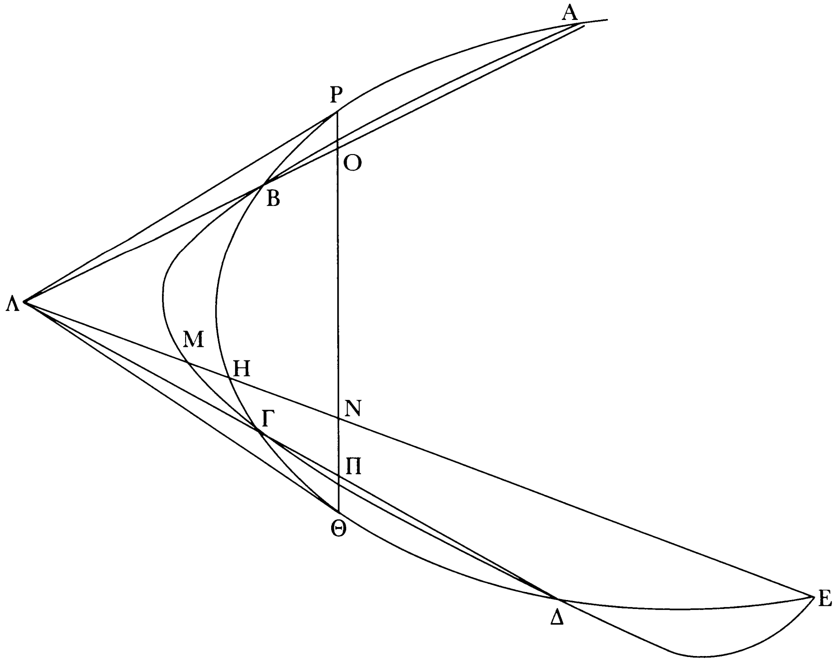

Fig. 10

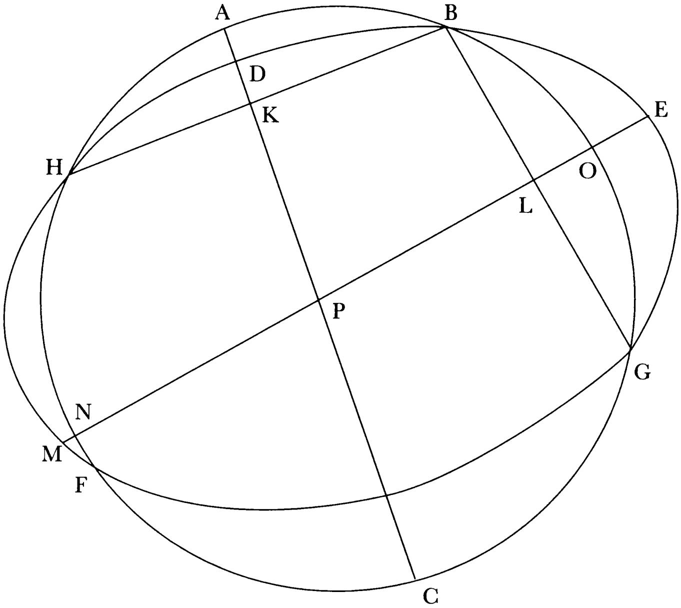

Fig. 11

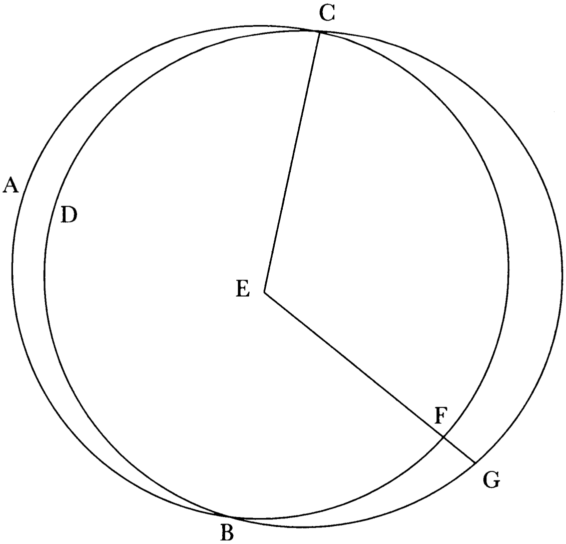

Fig. 12

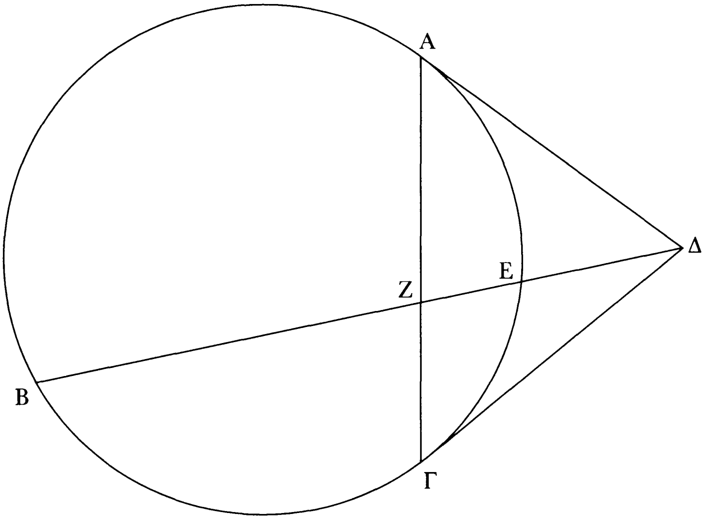

Fig. 13

CHAPTER EIGHT

THE *CONICA* AS AN INTEGRAL WHOLE: ELEMENTARY AND NON-ELEMENTARY BOOKS

Introduction

Although in previous chapters, and especially in the last, we have considered connections among the different books of the *Conica*, on the whole, our approach has been to take each book separately and examine the geometric character of its motivation and development. This was necessary, among other things, to counter non-geometric interpretations of these books and to show, therefore, that without importing ideas from algebra and analysis, these books still have (or have even more) cogency. However, even as we looked at the flow of ideas in the various books of the *Conica* and the sense of each book *by itself*, it was *never* our goal to paint a picture of the work, or even only the second half of it, as a collection of independent treatises. So now, in this chapter, we need to go back and ask in what way can one look at the *Conica* as a *whole* book. Indeed, we need to ask whether it is right to call the *Conica* a *whole* book at all. The conclusion, not surprisingly, will be that it *is* right. Yet, this does not mean that the question is trivial, nor that the answer is completely unequivocal. The problem is that, on the one hand, the only indubitable evidence of the *Conica* being a whole is that Apollonius refers to it as a whole, while, on the other hand, looking at the seven extant books together it is not easy to find precisely their coherence. In the course of the chapter, we shall concentrate on what can be said about this issue from the prefatory letters to the various books and especially from Apollonius's remarks in the preface to Book I. As for the latter, we shall give most of our attention to the division Apollonius makes between the first and second four books, the books Apollonius calls "elements" and those about which he says that they are "fuller in treatment" than the first four. There will be in this, unavoidably, some overlap with what was said in our chapter on the elements of conic sections in the framework of Books I to III and in our chapter on Book IV. But whereas the main object

of those chapters was to explore the nature of Apollonius's elements, his starting points, the emphasis here is to show what, if anything, distinguishes the elementary from the non-elementary books, and to see how the two parts form an integral whole.

Was the Conica *Written as One Book?*

The letter prefacing Book I of the *Conica* has been a continual point of reference throughout this work. This is simply because that letter, unlike the corresponding letters opening Books II, IV, V, VI, and VII, was written not as an introduction to Book I only, but to *all* of the *Conica*. Hence, Heath refers to the letter, correctly, as Apollonius's "General Preface."[1] This very fact, then, constitutes our first and main evidence that Apollonius thought of the *Conica as one book*. Yet, it is in connection with this same "General Preface," that one also raises one's first doubts as to whether the *Conica* really is one book. Let us hear how Apollonius begins this letter.

> Apollonius to Eudemus, greetings.
> If you are restored in body, and other things go with you to your mind, well and good; and we too fare pretty well. At the time I was with you in Pergamum, I observed you were quite eager to be kept informed of the work I was doing in conics. And so I have sent you this first book revised (διορθωσάμενος), and we shall dispatch the others when we are satisfied with them. For I don't believe you have forgotten hearing from me how I worked out the plan for these [conics] (διότι τὴν περὶ ταῦτα ἔφοδον ἐποιησάμην) at the request of Naucrates, the geometer, at the time he was with us in Alexandria lecturing, and how on arranging them in eight books (πραγματεύσαντες αὐτὰ ἐν ὀκτὼ βιβλίοις) we immediately communicated them in great haste because of his near departure, not revising (οὐ διακαθάραντες) them but putting down whatever came to us with the intention of a final going over. And so finding now the occasion of correcting them, one book after another, we publish them (ἀεὶ τὸ τυγχάνον διορθώσεως ἐκδίδομεν). And since it happened that some others among those frequenting us got acquainted with the first and second books before the revision (πρὶν ἢ διορθωθῆναι), don't be surprised if you come upon them in a different form (ἑτέρως ἔχουσιν).

[1] For example, Heath, *Apollonius of Perga*, p. lxix.

That Eudemus has received the first book of the *Conica* together with this letter is alone what justifies the letter being specifically a preface to Book I; in the rest of the letter, Book I really receives no more attention than any other book. Moreover, the reason Eudemus has received *only* the first book is that that was the only book Apollonius had managed to revise thus far. In this sense, therefore, the final revised edition *Conica* was not yet a finished book when Apollonius wrote his letter to Eudemus. However, instead of receiving at once a draft of the whole book, Eudemus was to receive the book piecemeal. Accordingly, Eudemus received the first book, then the second, then the third. As we know, Eudemus passed away before he could receive the fourth book. The pattern, however, continued, for now a certain Attalus[2] received the remaining books, Book V, VI, VII, and, presumably, Book VIII, book by book. Indeed, not only was each book *sent* separately, each was also revised (διορθωσάμενος) separately.

Apollonius refers to the process of revising books of the *Conica* four times in the passage quoted above from the prefatory letter. In one of the these four instances, he speaks of sending the books to Naucrates unrevised (οὐ διακαθάραντες), meaning, literally, "not thoroughly purged" or "cleansed." Apollonius's use of the verb διακαθαίρω here suggests that, in this instance, by "revising," Apollonius means "correcting." As for the rest, he uses the verb διορθόω meaning "to make quite straight, set right, amend," as its connection with the word ὀρθός would imply. It is plausible that Apollonius prefers διορθόω because the kind of revisions he is mostly speaking about are not those involving correcting the text as such but revisions involving ordering and clarifying the text, straightening it out, as it were.[3] This

[2] Nothing definite, to our knowledge, is known about Attalus. Toomer, however, argues convincingly that this Attalus cannot be King Attalus I of Pergamum, who reigned in Pergamum at the time Apollonius possibly was writing the *Conica*, for, if King Attalus were intended, surely Apollonius would have referred to him using the word "King" (βασιλεύς) which he does not (see Toomer, "Apollonius of Perga," p. 179). Toomer elsewhere has "speculated whether Apollonius' addressee is to be identified with the mathematician Attalus of Rhodes (known mainly from Hipparchus' criticism in his commentary on Aratus), but apart from chronological difficulties, Ἄτταλος is too common a name for any certainty" (*Apollonius Conics Books V to VII*, p. xii, n. 2).

[3] We, thus, go slightly farther than Jaap Mansfeld who writes, "The preliminary version [of the *Conica*] therefore cannot have been very rudimentary. Revision must have been a matter of style, of adding prefaces, etc." (*Prolegomena Mathematica*, p. 37).

is supported by the fact that Apollonius never refers to anything in the unrevised versions as *wrong*, and, with regards to the earlier versions of Books I and II at least, Apollonius says only that Eudemus might find them *different* (ἑτέρως) from the revised versions. The kind of revisions implied by the word διορθόω seems to be, therefore, referring to revisions that turn a farrago of results into a book. Thus, while we cannot know precisely what were Apollonius's revisions, it is fair to assume that they were such that the individual books finally sent to Eudemus and Attalus were, to Apollonius's satisfaction, not only free from errors but also well organized and readable—complete books.[4] One might argue that the books of the *Conica* being sent one by one, and fully revised, gives credence to a view of the *Conica* as a collection of quasi-independent treatises, for if the *Conica* were a single well organized whole one could well imagine that revisions in a later part might demand revisions in an earlier part. As a concrete illustration of such an instance, one could consider the relationship between IV.24 and VI.6,10. IV.24 states that "A conic section cannot meet [another] conic section or circumference of a circle in such a way that one part is and [another] part is not common," while VI.10 (and VI.6 contrapositively) states "[In] sections which are unequal, no part of one of them will fit on a part of another." Now, IV.24 is not used subsequently in Book IV, nor is it referred to in the proofs of VI.6,10 which are only restatements of IV.24, as was pointed out in the last chapter. Moreover, while IV.24 is not completely out of place in Book IV, VI.6,10 seem much more in line with the material of Book VI, so that a hypothetical editor of the whole *Conica* would certainly be justified in *eliminating* IV.24 in the light of VI.6,10.[5]

An unqualified view of the *Conica* as a collection of quasi-independent treatises, however, cannot be true. Strictly speaking, it is correct to

[4] On the need to write books that are readable, one might consider the following remark by G. E. R. Lloyd: "Although such mathematicians as Archimedes of Syracuse and Apollonius of Perga wrote principally for other mathematicians, they sometimes dedicated their treatises to notable heads of state, doing so in some cases not merely out of tact, *but in a way that shows they expected their work to be understood* [emphasis added]" (Lloyd, *Greek Science After Aristotle*, p. 5).

[5] Of course, we are not claiming, in this particular case, that Apollonius necessarily *should* have carried out such a revision (there may be many good reasons why Apollonius did not eliminate IV.24, and, there is always the possibility that one of the propositions IV.24, VI.6,10 was a later interpolation), only that this is the kind of case in which revision would have been in order.

say that Apollonius's work on conics had not been completed when he sent the first book to Eudemus. But it had not been completed only in the sense that it had not yet been revised. We know that the bulk of the work on conic sections had already been done when Eudemus received the first book, for Apollonius previously sent the eight unrevised books to Naucrates; the individual books were only being revised and sent one by one, not written one by one. In fact, it might even be that these individual books were being sent out relatively quickly; Apollonius says, for example, in the prefatory letter to Book VII that he will "strive to send [the last book] to [Attalus] speedily."[6] And, that Eudemus passed away before Apollonius could send him all the books is no evidence that the process of publishing the books was necessarily a slow one.[7] Eudemus may very well have been old, sick, or both when Apollonius communicated the first three books to him—one notes, indeed, that in the two letters to Eudemus, Apollonius asks about Eudemus's health, while he does not show a similar interest in Attalus's health. But whether the revision process was long or short, the important point to reiterate is that before the process of revision had begun and before he sent the final version of the various books to Eudemus and Attalus, Apollonius had already carried out a complete investigation of conic sections in Alexandria and set it out in eight books. This we are told not only in the passage from the prefatory letter to Book I, but also in the prefatory letter to Book IV, where Apollonius tells Attalus that, earlier, he sent Eudemus the first three books of his "[work] on conics organized together into eight books (τῶν συντεταγμένων . . . κωνικῶν ἐν ὀκτὼ βιβλίοις),"[8] that is, *already* organized into eight books.

The word συντεταγμένων, which Apollonius used in the passage above to describe the organization of the work he has written on conic sections, brings to mind another such "organized" work, though

[6] As throughout this work, translations from Books V to VII are from Toomer's *Apollonius Conics Books V to VII.*

[7] Heath seems to imply that the publication *was* a taking a long time by translating the word πρότερον, which opens the prefatory letter to Book IV, as "Some time ago . . ." (*Apollonius of Perga*, p. lxxii). We do not see why one cannot translate πρότερον more literally as "first," "before," or "earlier" so that the letter would begin, "Earlier, I sent the books to Eudemus . . ., but [he] having passed away, I send the remaining books to you . . ."

[8] Apollonius uses the same expression also in the letter to Eudemus that accompanies Book II: "I have sent you my son Apollonius bringing you the second book of the conics as arranged by us (. . . τὸ β′ βιβλίον τῶν συντεταγμένων ἡμὶν κωνικῶν)."

a much later one, Claudius Ptolemy's Σύνταξις. In general, the verb συντάσσω implies not just throwing things together but systematically organizing things together, putting soldiers in an array,[9] for example. The words Apollonius uses in the prefatory letter to the first book when he speaks of "... arranging [the work on conics] in eight books (πραγματεύσαντες αὐτὰ ἐν ὀκτὼ βιβλίοις) ..." also carries overtones of a systematic investigation. Besides basic meanings of "to busy oneself" and "to treat laboriously," Liddell and Scott say that πραγματεύομαι among historians (and apparently not only amongst them) means "to treat systematically," and cite Polybius as a source.[10] Indeed, in another place, Polybius makes it quite clear in his *Histories* what such a systematic work means for him. He writes, "For since my design is to write the history not of certain particular matters but of what happened all over the world, and indeed, as I previously stated, I have undertaken, I may say, a vaster task than any of my predecessors, it is my duty to pay particular attention to the matter of arrangement and treatment, so that both as a whole and in all its details my work may have the quality of clearness."[11] In this light, then, the *Conica*, like Polybius's *Histories*, can plausibly be seen as another fruit of the general Hellenistic tendency not only to advance but also to systematize knowledge.[12]

To conclude this section, let us try and put together what has been said so far and propose a possible course that the writing of the *Conica* might have followed. First, no doubt being aware of work being done by others on conic sections[13] and, very likely, of work by Apollonius himself, the geometer Naucrates requests that Apollonius take up the subject in a *systematic* way. We can assume that this was truly Naucrates' request, since it was a single work arranged in eight books, or at least a draft of such a work, that Apollonius presentes to Naucrates on the latter's departure from Alexandria. That Apollonius

[9] This is the *first* possibility under συντάσσω in *LSJ*.

[10] Polybius, *Histories*, V.33.

[11] *Ibid.* V.31. We have taken advantage here of W. R. Paton's English translation in vol. III of *Polybius, The Histories* (6 vols.; Cambridge: Harvard University Press, 1954), p. 79.

[12] See, for example, M. Smith, *The Ancient Greeks*, pp. 123–124.

[13] Work on conics before Apollonius has been well documented throughout Zeuthen's *Die Lehre*, chaps. I–III of Heath's *Apollonius of Perga*, chap. III of Dijksterhuis' *Archimedes*, and Toomer's "Apollonius of Perga," among others. Of course, all of these modern accounts only supplement the many references by Apollonius himself to previous work in conics.

arranges the results in this systematic form *hurriedly* suggests that in any other form Apollonius might not have fulfilled Naucrates' expectations. Of course, we have no idea how much work on conics Apollonius did before Naucrates' visit, or whether any work that he did do was done with an eye to making a systematic investigation. But whatever the case, with Naucrates' request Apollonius would have had to think about his results on conic sections, at one point or another, as parts of a single systematic treatise. It was *within the framework of such a single treatise* that Apollonius now began to revise individually the eight books of the *Conica.* So, to return to a point we made earlier, Eudemus does not receive a draft of the whole of the *Conica* because such a draft already exists, namely, the version passed on to Naucrates. Indeed, it may be the knowledge of this version in Naucrates' possession (perhaps because Apollonius spoke about it when Eudemus was with him in Pergamum) that made Eudemus "quite eager to be kept informed of the work [Apollonius] was doing in conics." In any case, since there exists already a draft of the whole work, it is now for Eudemus, and afterwards for Attalus, to receive the finished revised versions of the individual parts on which Apollonius had presently begun to work.

Elementary and Non-Elementary Books: The Basic Division in the Conica

Knowing that Apollonius wrote the *Conica* as a whole book still does not answer what is truly the more important question, namely, what makes the *Conica* a whole. Immediately following the passage that we have been discussing from the letter accompanying Book I, Apollonius furnishes us with a description of what each book of the *Conica* is about; it is like an expanded table of contents.[14] But a table

[14] For this reason, presumably, Pappus (*Coll. VII*, 674.20 (Hultsch)) describes this part of Apollonius's letter as a "summary prospectus" (Jones' translation of κεφαλαιώδη προδήλωσιν). Referring to this part of the letter, Mansfeld remarks that it is "Interesting not only because Pappus correctly calls the dedication/introduction a 'proem', but also because he calls this summary of the contents a κεφαλαιώδη προδήλωσιν, a 'preliminary heading-like *clarification*', i.e. one listing in a clear way the main *themes*. The substantive προδήλωσις ('announcement', 'prediction', cf. the meaning the verb usually has) is very rare—in the *Collectio* it occurs only here—, and its present meaning is not listed in the new LSJ. The formula as a whole is an excellent designation of what an introduction should contain in the matter of a listing of topics" (Mansfeld, *op. cit.* p. 39).

of contents by itself does not show, necessarily, how the various chapters and sections of a book are put together. As for the general structure, or rationale, of the whole book, Apollonius gives us only one hint, namely, the division of the *Conica* into four books of elements and four books "fuller in treatment." But how informative is this really? It does, of course, provide further evidence for Apollonius's thinking of the *Conica* as a whole book, both in the mere evenness of the division and in the fact that four books are thought, in some way, to set the stage for the other four books. In this connection, it can be observed that in Euclid's *Elements* too the first four books have a character setting them apart from the remaining books. On the other hand, the exact basis for the division in Apollonius's work is not clear in all its details. Certainly, deductive structure gives a partial answer, for each of the Books V, VI, and VII depends heavily on Books I–III. However, this is not true with regards to Book IV; its results are not used in the subsequent books. Why is Book IV included in the first part of the division? Why not have Book VI end the "course in the elements"? Are not equality and similarity elementary issues in every sense of the word? Now, one cannot reject completely the possibility that the division is simply a consequence of logical dependence, on the one hand, and the desire to have an even "four plus four" division, on the other, so that there is some arbitrariness involved. For an advanced mathematical work, one naturally hopes (though it may turn out to be a vain hope) that the ordering of the whole did not rest on the flip of a coin. Let us consider, therefore, some other possibilities.

The first characterization, and probably the most commonly accepted one, regarding the division between Books I–IV and Books V–VIII is that the first four books go over known material while the latter four "break new ground," to use Heath's phrase.[15] In his book on Apollonius, Heath writes, "Thus the first four Books [of the *Conica*] were limited to what were considered the essential principles; and their scope was that prescribed by tradition for treatises intended to form an accepted groundwork for such special applications as were found e.g. in the kindred theory of *solid loci* developed by Aristaeus. It would follow that the subject-matter would be for the most part the same as that of earlier treatises, though it would naturally be

[15] Heath, *Apollonius of Perga*, p. lxxvi.

the object of Apollonius to introduce such improvements of method as the state of knowledge at the time suggested, with a view to securing greater generality and establishing a more thoroughly scientific, and therefore more definitive system."[16] Toomer, in his article on Apollonius in the *Dictionary of Scientific Biography* takes this position further, though, again, not without his own reservations. Thus, Toomer writes, "From [the preface to Book I] one gets the impression that Books I–IV, apart from the subjects specifically singled out as original, are merely reworkings of the results of Apollonius' predecessors. This is confirmed by the statement of Pappus, who says that Apollonius supplemented the four books of Euclid's *Conics* (which Pappus *may* have known) and added four more books."[17] A few lines later, having discussed Apollonius's treatment of the opposite sections as a possible innovation, Toomer concludes, "But we are not justified in assuming that any of the results stated in the first four books were unknown before Apollonius, except where he specifically states this. In this part of the work we must see him rather as organizing the results of his predecessors, consisting in part of haphazard and disconnected sets of theorems, into an exposition ordered rationally according to his own very general method."[18] By contrast, we have ". . . Books V–VII, which are, to judge from Apollonius' account, largely original . . ."[19]

The problem with both Heath's and Toomer's account is that, while both are careful to qualify their statements, they take the qualification as *mere* qualification; they seem to take the fact that there are in Apollonius's account "subjects [in the first four books] specifically singled out as original" too lightly. Indeed, "to judge from Apollonius' account," there is about as much original and not original material in Books I–IV as there is in Books V–VII. Consider, first, the statements in the prefaces to Books I–IV in which Apollonius

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[16] *Ibid.* Heath's statement is, as one can see, a qualified one. Later, he further qualifies the point by emphasizing Apollonius's originality in the first four books: "At the same time it seems clear that, in the generality of [Apollonius's] treatment of the subject from the very beginning, Apollonius was making an entirely new departure" (*Ibid.*, p. lxxviii). We accept this statement. Our disagreement with Heath here has only to do with the weight that we think should be put on the statement. If the work in the first books is "an entirely new departure," then one should not see them as only going over or reorganizing old and known material.

[17] Toomer, "Apollonius of Perga," p. 185.

[18] *Ibid.*, pp. 185–186.

[19] *Ibid.*, p. 187.

says that he is doing something new. Regarding Book I, Apollonius says that that book "contains the generation of the three sections and of the opposite branches, and the principal properties in them *worked out more fully and universally than in the writings of others* [emphasis added]." Apollonius also says quite explicitly in the preface to Book I, that "The third book contains many incredible theorems of use for the construction of solid loci and for limits of possibility of which the greatest part and the most beautiful are new." In the preface to Book IV too, he mentions, regarding the number of ways a conic section can meet the opposite sections, that he ". . . has found no demonstration either by [Nicoteles] or by anyone else," and, regarding the ways in which the opposite sections can meet one another and the like, that he has ". . . not found them even noticed by anyone."[20] At the same time, in the prefatory letters for Books V–VII, one does not see Apollonius much *more* emphatic about the newness of what he demonstrates in these latter books. In the preface to the fifth book, Apollonius only alludes to his treatment of maximum lines and the general thoroughness of his treatment of the whole subject of maximum and minimum lines as new. He writes, "You should realize that our predecessors and contemporaries paid (a little) attention only to the minimum [lines] . . ." and, a little later in the preface, "As for the propositions in which we speak of the minimum lines, we have separated them out and treated them individually, after much investigation, and have attached the discussion of them to the discussion of the maximum lines which we mentioned above . . ." In the preface to Book VI, Apollonius only says that "What we have stated on [the subject of the equality and similarity of conic sections][21] is fuller and clearer than the statements of our

[20] Apollonius describes in the prefatory letter what problems are his concern in Book IV: "This [book] treats of the greatest number of points at which sections of a cone can meet one another or meet a circumference of a circle, assuming that these do not completely coincide, and, moreover, the greatest number of points at which a section of a cone or a circumference of a circle can meet the opposite sections." Following this, in the passage to which we referred above, Apollonius, however, speaks only of "the second [question]," and "the third [question]." Heath defends Halley's interpretation that the first problem is "the intersections of single-branch conics with one another or with a circle," that the second is "the intersections of a single-branch conic or a circle with the double branch hyperbola," and the third is "the intersections of two double-branch hyperbolas" (Heath, *Apollonius of Perga*, pp. lxxii–lxxiii). We accept this interpretation, mostly because it is consistent with the order in which Apollonius proceeds in the book itself.

[21] Our insertion, not Toomer's.

predecessors." And in the preface to Book VII, he echoes his description of the new results set out in Book III, saying "In this book [Book VII][22] are many wonderful and beautiful things on the topic of diameters and the figures constructed on them, set out in detail." So, whatever may be the true distribution of originality and unoriginality in the *Conica*, there is little *in the letters* prefacing the various books showing that, in Apollonius's estimation at least, the first four books were distinguished from the second four books in this respect.

One cannot help agreeing with Heath when, on the subject of Apollonius's work in Book I being carried out "more fully and universally than in the writings of others," Heath writes, "So far from being a braggart and taking undue credit to himself for the improvements which he made upon his predecessors, Apollonius is, if anything, too modest in his description of his personal contributions to the theory of conic sections. For the 'more fully and generally' of his first preface scarcely conveys an idea of the extreme generality with which the whole subject is worked out."[23] But since this, therefore, must apply equally to the passage in the preface to Book VI, quoted above, the claims of originality in Books I and VI, at least, must be given the same weight. Indeed, it can be argued that Books I–IV are *more* original than the others. For example, Apollonius's account of Books I–IV in the "General Preface" is far longer and detailed than his account of Books V–VIII, as we mentioned in the first chapter to show that there was no reason to think Apollonius expected Eudemus to be less interested in the "course in the elements" than in the remainder of the book. If the other books were the more original ones, one would expect Apollonius to describe them more expansively and to stress their originality at least as much as he does the results in Book III. The same can be said too concerning the part of Pappus's *Collection* Toomer uses to confirm that the contents of Books I–IV ". . . are merely reworkings of the results of Apollonius' predecessors." It is true that Pappus claims "Apollonius filled out Euclid's four books of *Conics* and added on another four . . ."[24] One would expect such a remark to open a discussion in which

[22] Again, our insertion.
[23] Heath, *History*, II, p. 133.
[24] *Coll. VII*, p. 672 (Hultsch). Here, as always, we are taking advantage of A. Jones' excellent English translation in his *Pappus of Alexandria: Book 7 of the Collection.*

these books would be largely ignored in favor of the "more original" books V–VIII. Yet, in fact, almost all of Pappus's remarks about the *Conica* relate *only* to books I–IV. Moreover, Pappus clearly accepts Apollonius's priority in the discovery that all the conic sections can be generated from a single cone. On the other hand, most of Pappus's remarks are dedicated to defending Euclid's attempts at demonstrating the three and four line locus. On this, Pappus says (being rather less willing to praise Apollonius's modesty than was Heath) that Apollonius "was able to add the missing part to the locus because he had Euclid's writings on the locus already before him in his mind, and had studied for a long time in Alexandria under the people who had been taught by Euclid, where he also acquired this so great condition (of mind), which was not without defect. The locus on three and four lines that he boasts of having augmented instead of acknowledging his indebtedness to the first to have written on it, is like this . . ."[25] Despite Pappus's accusing Apollonius of treating Euclid shabbily, however, Pappus refrains from accusing Apollonius of not demonstrating what he said he demonstrated. But more than this, one can argue that Pappus's concentration on the priority of work connected with the three and four line locus only shows that more was at stake in being original here than in the other subjects represented in Apollonius's work on conics.[26] In other words, while Pappus, himself, may not have accepted Apollonius's originality in the treatment of the locus problem, the very polemic tone of his discussion may well show that Pappus's view on this was not the established view. Let us not, however, push this too far; what is at issue here is not that the first four books are *more* original than the second four, but only that there is little evidence to say that they are *less* original.

The next possible criterion for distinguishing the first and second four books of the *Conica* has to do with problem-solving. The comparison here is something like that between Euclid's *Elements*, on the one hand, and Euclid's *Data* and *Porisms*, on the other. Euclid's *Elements* has a different character from his *Data* and, from the little

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[25] *Coll. VII*, p. 678 (Hultsch).

[26] Naturally, there are other possibilities, such as the fact that the other party in the priority dispute happens to be Euclid, who is the other main player in Book VII of the *Collection*, or Pappus's own interest in the three and four line locus and its generalization, which he states in the same passage (p. 680).

one can gather from Pappus's treatment of it, from his *Porisms*; unlike the latter two works, the *Elements* "scarcely reflects the intense contemporary interest in problems of construction, and the expansion of the geometer's resources to tackle them," as Jones has said.[27] So, do the books that Apollonius called "a course in the elements" have this same relationship with the remaining books of the *Conica* that Euclid's *Elements* has with the *Data* and *Porisms*? Are the last four books marked by their emphasis on problem-solving? Knorr, for one, thinks they are. As mentioned (and criticised) in the last chapter,[28] Knorr sees an emphasis on problem-solving throughout the *Conica*, but, as he says, ". . . the problem-solving aspect is especially prominent in the later books, treating the more advanced portions of the theory."[29]

Now, it is incontrovertible that there is *an* interest in problem-solving in the later books, even a strong interest. Our own chapter on Book V stressed some connections between Apollonius's work on maximum and minimum lines and the problem-solving effort, and, of course, Apollonius's preface to Book V points out that book's usefulness for the ". . . analysis and determination of problems and their synthesis . . ." quite explicitly. Again, in the preface to Book VII, Apollonius says that the results in Book VII are ". . . of great use in many types of problems, and there is much need for it in the kind of problems which occur in conic sections which we mentioned, among those which will be discussed and proven in the eighth book of this treatise." Indeed, the subject of the eighth book, according to Apollonius's description in the preface to Book I, is "determinate conic problems (προβλημάτων κωνικῶν διωρισμένων)." Although Book VIII is lost, Halley's thoughtful reconstruction of it does, we think, give a very plausible view of what the book might have been like.[30]

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[27] A. Jones, paraphrasing Knorr (for example, *The Ancient Tradition of Geometric Problems*, p. 102), in "Greek mathematics to AD 300," p. 48.

[28] Note 6.

[29] Knorr, *Ancient Tradition of Geometric Problems*, p. 313.

[30] The other notable reconstruction of Book VIII is Ibn al-Haytham's *Completion of the Conics* composed sometime before 1021. About these two reconstructions, J. P. Hogendijk, who has produced a critical edition and English translation of al-Haytham's work (*Ibn al-Haytham's Completion of the Conics*), writes, "One might well ask which of the two reconstructions is the better one (i.e. the closer to Apollonius), the one by Halley or the one by Ibn al-Haytham. The question is unfair because Halley knew more about Book VIII than Ibn al-Haytham, and the answer is largely a matter of taste. Halley's reconstruction has the advantage that all (or at least

In his reconstruction, one finds problems such as "Given the figure on the axis of a hyperbola and given another diameter in magnitude: determine the position of the diameter, the position and magnitude of its conjugate, and its *latus rectum*" (VIII.5), "Given the axis and latus rectum [to the axis] of a hyperbola, find conjugate diameters, in magnitude and in position, containing a rectangle equal to a given rectangle" (VIII.13), and "Given the axis and latus rectum of an ellipse; find diameters whose squares exceed the squares of their latera recta by a given difference, and by the same fall short" (VIII.33). The character of these both as problems concerning conic sections themselves as well as ones potentially useful in other "solid problems," is clear. It should also be pointed out that, on the whole, Halley's reconstruction is (unlike some modern interpretations) very faithful in style to other parts of the *Conica*.[31] Consider, as an illustration,

some) of its problems were somehow treated in Book VIII, probably as auxiliary constructions. It is doubtful whether any of the problems in Ibn al-Haytham's reconstruction were really in Book VIII. But on the whole, these problems are more exciting than the problems of Halley's reconstruction, so Ibn al-Haytham's reconstruction may be closer in level to the lost Book VIII" (p. 72). Elsewhere, Hogendijk writes "In view of the scope of the extant books of the *Conics* it is likely that Book VIII contained more than trivial elaborations of the preceding books. However, Halley's reconstruction is a rather trivial appendix to Book VII. Book VII gives not only all the necessary diorismoi, but also provides (in VII:6–20) clear indications how the problems in the reconstruction can be solved. This does not of course exclude the possibility that the subject-matter of the reconstruction formed part of the lost Book VIII, perhaps in the form of auxiliary problems. But Book VIII must have contained more than this" (p. 48). We tend to agree with Hogendijk that Halley's reconstruction probably represents only part of what Apollonius's Book VIII was in fact. This is both because of the arguments adduced in our chapter on Book VII above, and because, judging from Pappus's statement of the size of the whole work (Pappus says there were 487 diagrams or propositions in the *Conica*, whereas combining the number of propositions in Heiberg's edition of Books I to IV with the number of propositions in the Arabic version of Books V to VII, one obtains 387 propositions for Books I to VII), it is likely that Book VIII was more substantial than Halley's mere thirty three propositions. That said, in light of the example from Halley's work provided here and in our chapter on Book VII, to call the propositions in Halley's reconstruction only "trivial elaborations of previous books" seems to us an exaggeration; whether they are more or less "exciting" than those in al-Haytham's work is, of course, hard to say. We claim that Halley's reconstruction of Book VIII, while, perhaps, falling short of a *full* reconstruction, gives a reasonable view of the character of Book VIII, because of its style, its adherence to what Apollonius says about Book VIII, and the nature of its problems, which are in line with the rest of the text and seem to be of some use in solid problems.

[31] There are, however, a few somewhat un-Apollonian manipulations in Halley's reconstruction. For example, in the course of Halley's VIII.5, having demonstrated

Halley's VIII.4: "In a given ellipse, if any diameter is given with its *latus rectum*, we can exhibit the *latus rectum* of any other diameter.

> Let ellipse ΓΕΒΔ whose center is A be given [see fig. 1]; and let the *latus rectum* of diameter ΓΒ be given, and BZ, equal to half [the *latus rectum*], be placed along the extension of the diameter: moreover, let ΔΕ be any other diameter, and, through the three points E, B, Z, let a circle be described meeting the extension of DE at point H. I say EH is half the required *latus rectum.*
>
> For in an ellipse (by VII.30) the sum of the square on any diameter and of the figure of the same is always equal; that is, the rectangle contained by the diameter and by any other diameter together with its *latus rectum* is everywhere equal; and thus that contained by the halves of these [will everywhere be equal]:[32] therefore, the rectangle BAZ [that is, the rectangle contained by BA and AZ] is equal to that contained by EA and EA together with its semi-*latus rectum.* But (by *Elem.*III.36) rectangle EAH [that is, that contained by EA and AH] is equal to the rectangle BAZ; whereby AH is equal to the semidiameter EA taken together with its semi-*latus rectum*: for this reason AH exceeds EA by half the required *latus rectum*; therefore, twice the line EH is equal to that *latus rectum.*

Finding the *latus rectum* of a conic section, given other elements related to the section, is a problem that is not treated in other parts of the *Conica*, that certainly compliments earlier parts of the *Conica* (in particular, the sequence of propositions I.49–51), and that is of obvious importance in problems involving conic sections. In its solution, too, one is reminded by VIII.4 of other problems in the *Conica* such as II.47,48, which also make use of an auxiliary circle for their solution. From this example, and others we could give from Halley's reconstruction of Book VIII, it is fair to say that it is at least a reasonable guess of what Apollonius's Book VIII contained, and, thus, its character gives further grounding to a view of the *Conica* in which

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the proportion sq.BK:rect.ΝΞ,ΓΔ::ΞΜ:ΝΞ he continues, "quocirca quod sit sub ΝΞ & quadrato ex BK aequale erit contento sub ΝΞ & rectangulo sub ΓΔ & ΞΜ . . ." concluding from that that sq.BK equals the rect.ΞΜ,ΓΔ, and, therefore, that BK is a mean proportional between ΓΔ and ΜΞ. This sort of merry algebraic manipulation, needless to say, never appears in Apollonius's work. Such instances, though, are very few in Halley's reconstruction of Book VIII.

[32] Proposition VII.30 tells us that if AB and CD are diameters of an ellipse and KL and MN, respectively, are their *latera recta*, then sq.AB + rect.AB,KL = sq.CD + rect.CD,MN. From this, it follows (*Elem.*II.1) that rect.AB,(AB + KL) = rect.CD,(CD + MN), and, therefore, with ab = half AB, cd = half CD, kl = half KL, and mn = half MN, that rect.ab,(ab + kl) = rect.cd,(cd + mn).

". . . the problem-solving aspect is especially prominent in the later books."

The observation that problems have an important place in the later books, however, is, plainly, not enough to conclude that 1) problems have a *central* place in the later books and that 2) they are more prominent there than in the earlier books. As for the second point, except for the reconstructed Book VIII, it is not true that problems and problem solving are less prevalent in the earlier books than in the later. Books I and II both end with problems, and Book II contains, right at the start (II.4), the very important problem of describing a hyperbola passing through a given point and having two given lines forming an angle as asymptotes.[33] Indeed, when the synthesis of a problem requires the drawing of a conic section, the problems in I.52–60 and II.4 (which relies on I.55) most often provide the relevant constructions. As for Book III, one can again cite Pappus's great interest in the propositions at the end Book III relating to the problem of the three and four line locus, as well as Apollonius's own remarks, quoted above, on ". . . theorems [in Book III] of use for the construction of solid loci and for limits of possibility . . ." The preface to Book IV also points out the relevance of that book to the problem-solving enterprise, saying that

> The consideration of these [theorems] is of considerable use in the synthesis and determination of problems (ταῦτα δὲ θεωρηθέντα χρείαν ἱκανὴν παρέχεται πρός τε τὰς τῶν προβλημάτων συνθέσεις καὶ τοὺς διορισμούς).[34]

As for the second point, there does not seem to be very strong evidence that problem-solving is the *central* issue in the later books. Although in our chapter on equality and similarity we did find some relevance of the propositions in Book VI to problem-solving, we tried

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[33] In his "The Hyperbola-Construction in the *Conics*, Book II: Ancient Variations on a Theorem of Apollonius," Knorr holds that proposition 4 in Book II of the *Conica* is, in fact, an interpolation by Eutocius. However, Knorr does not dispute that the proposition itself is of Apollonian provenance.

[34] We are following here Toomer's example (cf. *Apollonius Conics Books V to VII*, pp. lxxxiv–lxxxv) and Heiberg's ". . . problematum . . . ad determinationes . . ." in translating τῶν προβλημάτων . . . τοὺς διορισμούς as "determination of problems." The meaning, however, is the same as in Heath's translation ". . . determinations of limits of possibility" (*Apollonius of Perga*, p. lxxiii). Ver Eecke's translation ". . . pour les constructions des problèmes et *pour leur discussions* [emphasis added]," (*Les Coniques D'Apollonius de Perge*, p. 282) in this instance seems to us less felicitous.

to show that the main thrust of the book was of a more general and theoretical nature. Indeed, the preface to Book VI says absolutely nothing about problems at all. And in the prefaces where Apollonius does mention problem solving, one does not get the impression that this is the *raison d'être* of the books prefaced. We would even go further and say that Apollonius tries actively *to avoid* giving this impression. Thus, having mentioned the importance of propositions in Book IV for problem-solving, Apollonius is quick to point out that, "Apart from the usefulness of these [propositions], they are worthy of acceptance through their very demonstrations; for, indeed, we accept many other things in mathematics for this and no other [reason]." Again, in the preface to Book V, Apollonius says that besides their use in problems, maximum and minimum lines ". . . are one of the subjects which deserve investigation in their own right." In general, despite Pappus's inclusion of the *Conica* among the classic works of analysis, these disclaimers by Apollonius, at least, show that writers such as Knorr go much too far in stressing the problem-solving aspect of the *Conica*. Statements such as "The *Conics* of Apollonius, for instance, *is by its own account ancillary* [emphasis added] to [the ancient problem-solving] effort, providing the essential introduction to the theory of conics through which the so-called 'solid' problems might be solved . . ."[35] and "Apollonius insists [!] that the theory of conics has value principally for its use in the investigation of geometric problems,"[36] accordingly, seem to us exaggerated. What one can gather from Apollonius's own account is that problem-solving in the *Conica* is important, but that it is not its most important aspect; problem-solving is present in different forms throughout the first four books and throughout the last four books, but it does not seem to have more of a place in the one part than in the other.

A distinction between Books I–IV and Books V–VIII based on what is *elementary* and what is *advanced*, in the ordinary sense of those words, should, if our second chapter on the elements of conic sec-

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[35] Knorr, *The Ancient Tradition of Geometric Problems*, p. v.

[36] *Ibid.*, p. 328. Knorr goes on to say "Moreover, [Apollonius] does indeed provide in his theorems a foundation suitable for the formal synthesis of such problems. Yet he rarely actually engages in the activity of solving problems as such, either in the *Conics* or in the other analytic works, as known to us." Is it not possible, then, indeed is it not likely that Apollonius "rarely actually engages in [this] activity" because he did *not* think that "the theory of conics has value principally for its use in the investigation of geometric problems"?

tions was at all successful, be viewed with suspicion. We shall not go over all the arguments we used to show that "elementary" must be understood, in this context, as "*fundamental*," and not "elementary" in the sense of being for beginners. However, we shall remind the reader that what makes the treatment of conic sections in the first books "fundamental," among other things, is its *closeness* to the geometric origin of the sections, its seeing the conic sections *as* sections of a cone. That this *does* play a part in the distinction between the first and second four books can be inferred from Apollonius's saying in the preface to Book V that, ". . . as for us, we have proven those things [concerning tangents to the sections][37] in Book 1 without making use, in our proof of that, of the topic of minimum lines; *for we wanted to make the place where those [things] were put near to our discussion of the derivation of the three sections, in order to show in this way that in each of the sections there may occur an infinite*[38] *number of properties and necessities of these things, as is the case with the original diameters* [emphasis added]."

It is significant that Apollonius mentions specifically the original diameter in this passage from Book V. The "original diameter" referred to here is, of course, the diameter *arising directly from the sectioning procedure of the cone*,[39] and, as we showed at length in chapter II, it is the closeness to the sectioning procedure, and not the simplicity of relations associated with it, that makes this diameter an *elementary* diameter. This is important, for one difference between Books I–IV and V–VII that strikes the reader of the *Conica* almost immediately is the overwhelming presence of the *axes* in the later books and the dominance of a "general" diameter in the early books. Indeed, after its general definition at the start of Book I, the axis appears explicitly in Books I–IV only at the end of Book II, in the problems II.46–53, and at the end of Book III, in III.43,45 (46–48), 49–52.[40] On the other hand, besides V.1–3, all the propositions in Book V that involve a diameter are given in terms of one of the axes. The same can be said about Book VI and Book VII where, again, there are few propositions that do not refer to the axes. About

[37] Our insertion.

[38] No doubt this was ἀπείρων, "indefinite," in the original Greek.

[39] Cf. Toomer, *Apollonius Conics Books V to VII*, p. 551.

[40] In III.46–48, the word, "axis," does not appear, but one knows that that is intended because Apollonius is relying on the same hypotheses as in III.45.

Book VII, a remark is in order here, for Toomer says that "The book is concerned with diameters, especially conjugate diameters, and almost all the propositions deal only with hyperbola and ellipse. In the course of the book Apollonius proves some remarkable theorems, notable the constancy of the sum of the squares on conjugate diameters for the ellipse (Prop.12), and of the difference of the squares on conjugate diameters for the hyperbola (Prop.13), and that the rectangle bounded by four tangents parallel to two conjugate diameters is of constant area (Prop.31)."[41] This description is correct, but, in one respect, misleading. Apollonius does not state the propositions cited in terms of the *constancy* of a sum, *constancy* of a difference, or *constancy* of an area, that is, he does not say, for example, that if in an ellipse AB is a diameter and CD is its conjugate and EF is a different diameter and GH its conjugate, then sq.AB and sq.CD together is always equal to sq.EF and sq.GH together. The constancy of sum, difference, and area, as pointed out by Toomer, can be inferred immediately from Apollonius's statements, but the statements themselves *are in terms of the axes*:[42] "[In] any ellipse, the sum of the squares on any two of its conjugate diameters whatever is equal to the sum of the squares on its two axes" (VII.12); "[In] every hyperbola, the difference between the squares on its axes is equal to the difference between the squares on any pair of its other conjugate diameters whatever" (VII.13); "When a pair of conjugate diameters is drawn in an ellipse or between conjugate opposite sections [i.e. hyperbolas],[43] then the parallelogram bounded by

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[41] Toomer, *Apollonius Conics Books V to VII*, pp. lxix–lxx.

[42] Immediately following the passage quoted above, Toomer goes on to say (p. lxx), "But these [props.12,13,31] are incidental to the main thrust of the book, which is to investigate the variation in and limits of diameters, including pairs of conjugate diameters, the associated *latera recta*, and various functions of both," again completely avoiding saying that the properties of diameters developed in the book *are invariably related to those on the axes.* Heath does a little better in this respect, saying "Apollonius is now in a position, by means of all these relations, resting on the use of the auxiliary points H, H', M, [the points on the *homologue*] to compare different functions of any conjugate diameters with the same functions of the axes, and to show how the former vary (by way of increase or diminution) as P [an arbitrary point on the section] moves away from A [the vertex relative to the axis]" (*History*, II,172). By insisting on the word "function," Heath puts too much stress on the notion of "variation," rather than *simply* the relating of properties on axes and properties on other diameters; this, we think, makes Heath's account less acceptable than Toomer's, and, to an extent, even less coherent.

[43] Toomer's insertion.

that pair of diameters with angles equal to the angles formed by the diameters at the center is equal to the rectangle bounded by the two axes" (VII.31). Thus, Book VII is not an exception to the pattern in the later books of working from the perspective of the axes.

If one ignores Apollonius's statement that he wanted to keep facts about conic sections in the books before Book V close to the geometric genesis of the sections, while still insisting that "elementary" is to be contrasted with "advanced," then this dominance of axes in the putatively advanced books, Books V to VII, is bound to seem paradoxical. For, forgetting the way in which the conic sections arise in Apollonius's cone, and the way in which a diameter arises at the same time, leaves one thinking of the diameters as if in a kind of hierarchy beginning with the axis, the geometrical relationships being particularly simple with respect to the axis; propositions given in terms of the axis, in this sense, should be "elementary," while those given in terms of any other diameter (in this setting, truly a "general" diameter), "advanced." The truth is that the "general" diameter ever-present in the early books is, as we have said more than once, not really "general" but the specific diameter that comes to be when the conic section is produced, and this diameter is not always an axis, as Apollonius is careful to point out after I.7. So, by following Apollonius faithfully, the paradox vanishes. But still a few more words need be said about the significance of the wide use of the axis in the later books.

First, however, in the context of the "course in the elements" the following can be said about the axis as a diameter. First, the original diameter of a section *can* be an axis; an axis is the original diameter, for example, in the constructions I.52,54,56. Second, given any conic section, one can, by II.46, always find its axis, which shows, incidentally, that every conic section *has* an axis. Finally, if the original diameter is not the axis, then I.49–51 demonstrate how the *latus rectum* for the axis can be found by means of the original diameter, or if one is *given* the axis how one can find the *latus rectum* of another diameter.[44] These facts allow one to present propositions from the

[44] We must emphasize that *we* are stating these in terms of the axis and original diameter; Apollonius only relates the diameter of a section and some other diameter. However, in almost all of the instances where Apollonius *uses* I.49–51, he begins with the axis and finds the *latus rectum* of another diameter. This is the case in I.53, I.55, I.58, VI.17, VI.18 and VII.5.

perspective of the axis, even when this is not the original diameter. By "allow" we mean specifically that the relations in such propositions can be traced back theoretically to the original relations inherent in the very sectioning of the cone. However, in terms of "closeness and farness," it is clear that *beginning with the axis* of a section places one in a position *farther* from the ". . . discussion of the derivation of the three sections," than one would be beginning with the diameter, *i.e.*, the *original* diameter, of the section. We think this is the reason why there are relatively few propositions in Books I to IV starting with the axis, while the opposite is true in the books following the books of elements. Having finished the "course in elements" with Book IV, Apollonius is no longer constrained to stay as close as possible to the origin of the sections and he can present the propositions in terms of the more convenient axis. Indeed, as if to remind one why he *can* present propositions in terms of the axis, there are places in the later books where one can see Apollonius actually moving from the axis to results related to another, perhaps the original, diameter.[45] One of these places is VI.17. This states, as the reader may recall, that "If there are similar sections, and tangents are drawn to them ending at their axes and forming equal angles with the axes, and diameters are drawn to the sections from the point of tangency, and a point is marked on each of those diameters, and the ratios of the lines between the points marked and the vertices of those diameters to the tangents are equal, and lines are passed through [each] marked point parallel to the tangents so that they cut off segments from the sections:

> then those segments are similar, and their position is similar.
>
> And if segments are similar and their position is similar, then the ratios of their diameters to the [corresponding] tangents are equal, and the angles which the tangents form with the axes are equal."

Here is the first half of the proof:[46]

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[45] One does see something similar in the constructions at the end of Book I, where the construction of a conic section having a given line as an axis is then used to construct a conic section having a given line as a diameter that is not an axis. There is the difference, though, that in these constructions the axis produced is the original diameter, whereas in VI.17,18, VII.5 the axis is not necessarily the original diameter.

[46] VI.17 takes up the case of segments of parabolas only; VI.18 covers the cases of the hyperbola and ellipse.

First . . ., let the similar sections be two parabolas, AB, KL [see fig. 2]; let their axes be AZ, KO, and the tangents to them ΓZ, MO.

Let angles AZΓ, MOK be equal.

We pass through points Γ and M diameters to the sections, namely ΓE, MΞ. Let (EΓ:ΓZ) equal (MΞ:MO).

We pass through points E and Ξ lines parallel to line ΓZ, MO, namely ΔB, NΛ [respectively].

Then I say that segments BΓΔ, ΛMN are similar and similarly situated.

Proof of that:

We draw from points A, K perpendiculars to the axes, namely AH, KP, [cutting ZΓ, OM in Θ, Π], and extend diameters EΓ, ΞM until they meet them [AH, KP] at points H, P.

We make (ΣΓ:2ΓZ) = ΘΓ:ΓH,
and (TM:2MO) = ΠM:MP.

Then ΣΓ and TM are *latera recta* for diameters ΓE, MΞ [respectively].

So sq.ΔE = rect.ΣΓ,ΓE,[47]
as is proven in Prop.49 of Book I.
And likewise sq.NΞ = rect.TM,MΞ.
And ∠KOM = ∠AZΓ:
and ∠KOM = ∠PMO,
while ∠AZΓ = ∠HΓZ,

because lines ΞP,EH are parallel to lines OK, ZA [respectively],

as is proven from Prop.46 of Book I.
"Therefore, ∠PMO = ∠HΓZ.
And the angles at points H and P are equal,
so ΔΘΓH is similar to ΔPMΠ,
and [hence] ΘΓ:ΓH = ΠM:MP.
"Therefore ΣΓ:ΓZ = TM:MO.
But (ΓZ:ΓE) had been made equal to (MO:MΞ).
Therefore, ΣΓ:ΓE = TM:MΞ.

Hence it will be proven, as we proved in Prop.11 of this Book, that, if lines are drawn to ΓE parallel to line ΔB, and lines are drawn to MΞ parallel to line ΛN,

and the ratios of these lines which are parallel to [the segment] bases ΔB, ΛN to the portions they cut off from the [respective] diameters adjacent to points Γ and M (which are the vertices of the sections [*sic*]) are equal,

and the ratios of the portions cut off from one of the diameters to

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[47] As previously noted, we deviate from Toomer's symbolic conventions in writing "sq." and "rect." instead of his "²" and "·".

those cut off from the other diameter are also equal, and the angles formed by the ordinates parallel to these two bases [ΔB,ΛN] and the diameter in both sections are equal (because the angles at points Γ and M are equal):

then segment BΓΔ is similar to segment ΛMN, and its position is similar to its position.

One sees very clearly here how Apollonius is using the axis as stepping-stone to speak about a segment whose diameter is not an axis.[48] This is done in two distinct ways, 1) through his use of I.49, and 2) through the analogy with the proof of VI.11, the proposition showing that every parabola is similar to every other. The first, of course, allows the precise correspondence between relationships with respect to the axis and relationships with respect to another diameter. As for the second, one must remember that the similarity of conic sections is defined in terms of the axis while similarity of segments is not. Apollonius might have shown that the segments are similar according to the definition, using the same argument as in VI.11, but without mentioning VI.11. The fact that he does mention this proposition, however, shows, we think, that he wanted to make a special point of the link between the similarity of sections and the similarity of segments. Moreover, the second part of the demonstration, that the similarity of similarly positioned segments leads to the similarity of the sections, further links similarity viewed from the perspective of axes with that of non-axis diameters.

Conclusion

From the prefaces to the various books of the *Conica*, we confirm that Apollonius views the book as a single systematic treatise, rather than an odd collection of works related only in their being all on conic sections. Apollonius sets out the plan of the treatise in the sense that he lists the subjects contained in each of the eight individual books of the *Conica.* He sets out the plan of the book *in the sense of giving some account of its structure*, however, only in his distinguishing the first four books as books of elements and the second four books as "fuller in treatment." Unfortunately, the exact basis

[48] The diameter of a segment, needless to say, is always along a diameter of the section by II.28.

for this division is somewhat elusive. We remarked that in this regard deductive structure alone does not appear to be enough, and, accordingly, we considered other possibilities, namely, the distinction between the first and second four books in terms of known and unknown results, in terms of the problem-solving enterprise, in terms of material for beginning and advanced students, and, finally, in terms of nearness and farness from the conic sections' geometric origins. None of these possibilities is completely satisfying. The last, however, has two things in its favor: 1) it is supported by Apollonius's remarks in the preface to Book V, and 2) it provides an explanation for one noticeable difference between the first and second four books, namely, the infrequent use of the axis in the early books and the predominance of it in the later books. This possibility is also completely in line with what we said about the nature of Apollonius's elements in Chapter II.

Does this distinction between the first four and second four books according to the nearness to and farness from the geometric origins of the sections tell us much about the whole of the *Conica*? It does and it doesn't. There are certainly many details in the fine, and even broad, structure of the book that it fails to explain adequately. However, it does imply that *both* parts of the work are being viewed in the light of the basic geometric constructions that christen the conic sections at the start of Book I. This observation, obvious though it may be, is of crucial importance for anyone who wants to understand the *Conica* as a whole, for it implies that the later books are dependent not only logically on the first books but also conceptually. In other words, what was set out in the early books not only provides results necessary for the demonstrations in the later books, but also for understanding what these demonstrations mean, what they are about.

Seeing the *Conica* as a *whole* book whose two main parts are related as we have set out above gives further strength to the conclusions reached in our chapters on Books I–IV. The arguments there showing the geometrical underpinnings of Apollonius's understanding of conic sections might be taken as representing only a *particular* view of conic sections that Apollonius entertained, while the other books, although they use some results from the former, might be taken to represent a completely *different* mode of thinking. Indeed, Apollonius *does* say in that same crucial preface to Book V that propositions demonstrated in the early books *could have* been demonstrated in a *different*

way within the context of the theory of maximum and minimum lines. But if, as we have been maintaining, the *Conica* is genuinely a whole book in which the first books are the foundation of the rest, then the geometric character of the first books must, at bottom, be *shared* by *all* the books, and the conceptual break involved in (artificially) separating the character of Books I–IV from that of Boois V–VII is totally unwarranted.[49]

[49] A certain historiographic objection might be made here, namely, that which Quentin Skinner called the "mythology of coherence." Skinner explains this as follows: "It may be (and indeed it very often happens) that a given classic writer is not altogether consistent, or even that he fails altogether to give any systematic account of his beliefs. If the basic paradigm for the conduct of the historical investigation has been conceived as the elaboration of each classic writer's doctrines on each of the themes most characteristic of the subject, it will become dangerously easy for the historian to conceive it as his task to supply or find in each of these texts the coherence which they may appear to lack." (Quentin Skinner, "Meaning and Understanding in the History of Ideas," p. 16.) Lynn M. Lindholm says something similar when he refers to a history of science that takes into account "confused" or "incoherent" positions (Lynn M. Lindholm, "A Hidden Inadequacy in History of Science," in *Scientific Philosophy Today*, J. Agassi and R. S. Cohen, eds., p. 163, also p. 184). This criticism by Skinner and Lindholm should, we think, be taken quite seriously and considered in any interpretive work in the history of science, especially one that takes into account different sources. However, at the same time, one must be careful not to take it to extremes and allow it, indeed, to stultify all such interpretive work. In this, we tend to agree with Kragh who writes that "A distinction should be made between the *assumption* of coherence and the *dogma* of coherence. While the latter invariably leads to bad historiography, the former can be a fruitful strategy when properly used. One should furthermore distinguish between the anachronical case, in which clarity is judged by modern standards, and the diachronical case, in which the assumption is defended without use being made of hindsight. Surely the historian is entitled to clarify obscure passages if he can justify the clarification by means of independent evidence. But he should not rule out the possibility that the text might, in fact, be obscure" (Helge Kragh, *Introduction to the Historiography of Science*, p. 100). We should add also that while Skinner and Lindholm's criticism is certainly relevant regarding the correspondences among different works of the same thinker, or a thinker's "worldview," as Lindholm puts it, or among the different thinkers comprised in a "school of thought," one should, generally, as a rule, be doubly cautious when applying it to a *single* work of a *single* author and concluding too easily and cavalierly that the author, in *this* context, is "confused." This, of course, is pertinent to the present discussion on the wholeness of Apollonius's *Conica*. Indeed, our discussion has been based on *Apollonius himself* referring to the *Conica* as a single work, and we were at great pains to show that *Apollonius* genuinely saw himself writing a single systematic treatise. Rejecting the legitimacy of asking what this meant to him, that is, what made the *Conica* a whole for Apollonius, would indeed be taking Skinner and Lindholm's criticism to an absurd extreme.

Figures

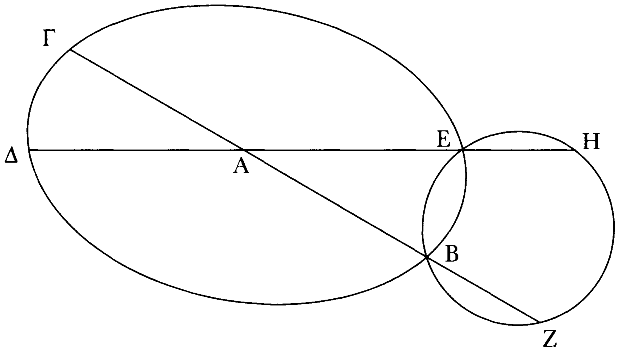

Fig. 1

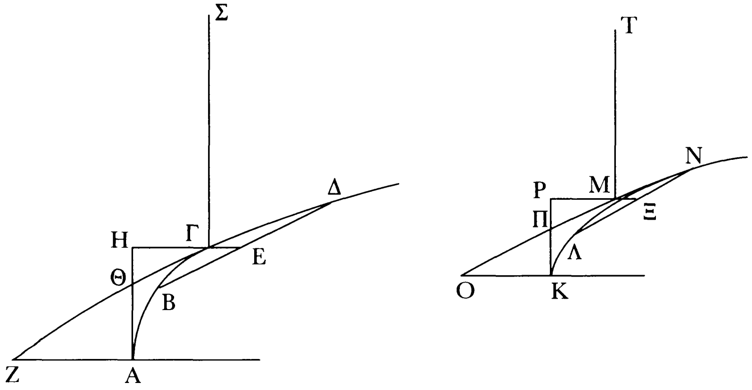

Fig. 2

CHAPTER NINE

SOME CIRCUMSCRIBED REFLECTIONS ON THE HISTORIOGRAPHY OF HELLENISTIC MATHEMATICS

In the historiography of Hellenistic mathematics, H. G. Zeuthen's *Die Lehre von den Kegelschnitten im Altertum* represents a summit of scholarship. Though its methodological approach is ahistorical, its author largely succeeded in pinpointing and identifying the main themes of Apollonius's treatise. Together with his other book-length study, dealing in part with ancient Greek mathematics, *Geschichte der Mathematik im Altertum und Mittelalter*, it is perhaps the best product of mathematically guided historical erudition. It is, however, neither the first nor the only study of conics or of Greek mathematics, despite its marking a turning-point in the serious study of the Greek mathematical legacy.

In this, the last chapter of our book, we wish to survey briefly some representative exemplars of historical learning about Greek mathematics, both preceding and succeeding Zeuthen, in order to reflect again on the main features of what counts traditionally as historical writing in our domain and, thus, set apart our contribution from all the rest.

One of the most distinguished historians of mathematics in modern times, who wrote a four-volumes treatise on the subject, which is still worth consulting, is J. F. Montucla.[1] His approach is more sensitive historically than that of most of his followers, though it is not immune to anachronism. Thus, though he reads the *Elements* algebraically, he is acutely aware of the differences between the synthetic, geometric style of the ancients and the analytic approach of the moderns (for whom he sees Descartes as typical)[2] and he gives an account of the *Conica* which is more attuned to the Greek way of doing things than that of Zeuthen.[3] All this does not mean, how-

[1] Cf. n. 55 in chapter II *supra*.

[2] *Op. cit.*, vol. I, pp. 68, 246, 248, 252, *passim*.

[3] *Ibid.*, pp. 169–171, 245–248.

ever, that Montucla's *Histoire des Mathématiques* is free of prolepsis. It is enough to look at his summary of the elementary properties of the sections in *Note* B, where he transcends freely Apollonius's treatment in the first four books, and to read his characterization of Book V, to convince oneself of the historically ambiguous character of Montucla's treatment.

Thus he points out correctly that Book V deals with maximum and minimum lines in conics but also (unjustifiedly, as we know from our detailed discussion of Book V, in Chapter IV above), that it contains "all that our present-day analytical methods teach us on the topic",[4] and that "one observes there the determination of our evolutes".[5] In short, even a solid and relatively sane study of Greek mathematics does not escape the lure of modern mathematical methods and notations in assessing and elucidating ancient mathematical texts. Indeed, as we shall see, using post-Greek and non-Greek mathematical insights becomes the generally accepted standard in dealing with Hellenistic mathematical works, in both original studies and in secondary textbooks.[6]

A classic of kinds in the encyclopedic treatment of the history of mathematics is Moritz Cantor's *Vorlesungen über Geschichte der Mathematik*.[7]

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[4] *Ibid.*, p. 247.

[5] The entire passage reads: "on y apperçoit enfin la determination de nos développées: car Apollonius remarque très-bien qu'il y a une suite de points dans l'espace au-delà de l'axe d'une section conique, d'où l'on ne peut tirer à la partie opposée qu'une ligne qui lui soit perpendiculaire. Il determine même ces points que les modernes connoissent sous le nom de *centres d'osculation*, et il observe que leur continuité sépare deux espaces, dont l'un est tel que de chacun de ces points on peut tirer deux lignes perpendiculaires à la partie opposée de la courbe, et l'autre au contraire a cette propriété qu'on n'en peut tirer aucune ligne semblable. Le premier de ces espaces est visiblement celui qui est renfermé entre l'axe de la courbe et sa developpée."

[6] Some ready, and relatively recent, examples of the latter (and these are the better-known and more accomplished instances) are, Howard Eves, *An Introduction to the History of Mathematics* (New York, etc., 1964), Carl B. Boyer, *A History of Mathematics* (New York, etc., 1968), Dirk J. Struik, *A Concise History of Mathematics* (New York, 1967), J. F. Scott, *A History of Mathematics* (London, 1960), H. Wussing, *Vorlesungen zur Geschichte der Mathematik* (Berlin, 1979), Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York, 1972), B. L. van der Waerden, *Science Awakening* (New York, 1963). Needless to say, this is just a spatter of the extant bibliography.

[7] Four volumes, some in later editions, published originally in Leipzig between 1892–1908.

The author, not a well-known working mathematician turning to history, as Zeuthen, say, typically refrains from mathematical technicalities in his discussion of historical topics, which often saves him from glaring anachronism. Still, the prevailing historiographic temper of the times does not turn its back on the *Vorlesungen.* This leads to a style of writing and historical analysis that has generally earned the scorn of more mathematically minded historians. Some examples, drawn from volume one, and dealing with the *Conica*, will make matters clear.

Curiously, in describing Book I, Cantor says nothing explicit about the *symptomata* propositions, 11–14, and, consequently, the typical equations introduced by Zeuthen and others in this context are not marring the discussion of the fundamental properties of the sections. Still, this does not prevent Cantor from speaking of true coordinate axes and a system of coordinates in Apollonius, which differ from "real coordinates," only in their not being independent, but only auxiliary, lines tied intimately to the specific sections.[8] "These given elements," Cantor goes on, "are handled by Apollonius in Greek fashion. Naturally, he does not compute by means of formulas and equations . . . but combines and links proportions of lengths and surfaces, which represent only another expression of the way of thinking embodied in the equations of the conic sections, to reach exactly the same conclusions."[9] This combination of sensible and anachronistic analysis is indeed characteristic of the entire Cantorian treatment.

Thus, in his description of the contents of Book III, Cantor describes all forty four first propositions as dealing "with ratios of products of tangents and secants to the conic sections," which is, strictly speaking, not true. He continues in the same vein, pointing out that, despite their great variety in the original, all these propositions can be summarized as two (!) main propositions, one dealing with the constant ratio between the product of segments of one secant and the product of the corresponding segments of the second secant drawn to the conic section from the same point, while the second synthetic proposition, involving secants and tangents, leads to a harmonic division of a certain segment in conics.[10] Now, it is well known that the concept of harmonic division is absent from the *Conica*, a

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[8] *Op. cit.*, vol. I, 4th ed. (1922), p. 337.
[9] *Ibid.*
[10] *Ibid.*, p. 338; see also, p. 341.

fact also known by Cantor himself, who points out, correctly, that Apollonius plainly "writes the proposition as a proportion"![11] Still, the charm of the modern, anachronistic concept is too great to resist.

The same mental attitude predominates in the remaining analysis of Book III, as well as in the assessment of the other books. It is clear that replacing Apollonius's rectangles and proportions with algebraic products and quotients is an acceptable and non-problematic procedure for Cantor, who also does not hesitate to speak of normals, in connection with some properties of lines drawn from "the points of application" (the modern foci).[12] He begins by approaching Book V sympathetically, describing its contents in terms of maximal and minimal lines, as one should, but, on the way, forgets his own care and slides down the dangerous ahistorical slope and speaks of subnormals, normals, centers of curvature, and evolutes, all missing concepts in the *Conica*, as we know.[13] Part of the guiding, schizophrenic, historiographic principle enabling this Janus-like approach seems to reside in the gratuitous belief, dating back at least to Descartes, that Apollonius employed secret methods for reaching his results, which he *post factum* proved artfully and artificially for the sake of publication:

> It is hardly possible to avoid suspecting, though one may not dare believe it, that Apollonius possessed some secret methods for discovering those propositions, the artificial proofs of which he only later appealed to.[14]

It is hardly possible to believe that this is followed by a sensible, lucid and largely unobjectionable account of Books VI, VII, and the reconstructive attempt of the lost Book VIII by Halley,[15] which ends

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[11] *Ibid.*

[12] *Ibid.*, p. 339.

[13] *Ibid.*, pp. 341–42. The language is particularly offensive: "Dann folgt eine Reihe von Sätzen, die etwa mit dem modernen Begriffe der Subnormalen sich beschäftigen. . . . Diese Punkte . . . sind dieKrümmungsmittelpunkte, deren Vorhandensein somit Apollonius bekannt war, so fremd ihm der Begriff der Krümmung geblieben ist. Möglicherweise ist es sogar nicht zu weit gegangen, wenn man annimmt, Apollonius habe die stetige Aufeinanderfolge der Krümmungsmittelpunkte geahnt, d.h., jene Kurve geahnt, wenn auch nicht untersucht, welche wegen anderer Eigenschaften der Namen der Evolute erhalten hat" (p. 342).

[14] *Ibid.*, p. 342: "Man kann kaum umhin zu argwöhnen, was zu glauben man doch nicht wagen darf, dass Apollonius irgend geheime Methoden besass, um diejenigen Sätze zu entdecken, deren künstliche Beweise er erst nachträglich aufsuchte."

[15] *Ibid.*, pp. 342–43.

with the following sound words: "These propositions too, derived understandably in a geometric, not a computational, way, demand from Apollonius to distinguish numerous individual cases, a procedure in which he reveals again the dexterity so much admired in the earlier books.[16] A mixture, then, or rather an *Eintopf*, a stew, of unexceptionable and objectionable ideas is what characterizes this non-technical, extensive treatment of the history of mathematics.

An altogether different exemplar of another multi-volume history of mathematics is represented by the 12 small-format tomes of Marie's aperçu.[17] Like Cantor, only more emphatically and carelessly, Marie thinks that the Greeks possessed a hidden algebra, by means of which they reached all of their significant results, which they later clothed in geometrical garments:

> The elements of algebra exist in Euclid's books, which his successors, who barely understood them, called *arithmetical*, though they contain the theory of ratios and proportions between concrete magnitudes; but the typical examples which we have extracted from Archimedes and Apollonius show clearly that these geometers have, in their practice, extended significantly this rudimentary algebra.
>
> Since, after having discovered analytically the most beautiful theorems, they thought it appropriate to provide for them *a posteriori* proofs which would be understandable without having to share in their creators' genius, and which it would have been impossible not to understand, provided one followed exactly their long chains of reasoning, no trace is preserved of the algebraic procedures by means of which they so marvelously transformed the relations which they encountered in their studies.
>
> Still, it seemed evident to us that absorbing fully their views, one can reconstitute that lost Algebra, which Viète would later call *speciosa*, as opposed to the Algebra *numerosa* of the period with which we are now dealing.[18]

Not much need be said about this anthological piece of reconstructive mathematical history, except to point out that Marie thinks, based on no historical evidence whatsoever, that the Greeks of Archimedes' and Apollonius's times were in possession of Vietan algebra, which supplied them with the (hidden) tools necessary for

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[16] *Ibid.*, p. 343.

[17] M. Maximilien Marie, *Histoire des sciences mathématiques et physiques* (Paris, 1883–1888); volume I, from which we shall quote, is entitled: *De Thales à Diophante*, contains 283 pages, its last chapter (pp. 262–283) being: "L'algèbre des géomètres grecs."

[18] *Op. cit.*, pp. 262–63.

their amazing mathematical discoveries! Since even he, however, cannot write the history of "a method which has not materialized in broad daylight [i.e., has not seen the light of day],"[19] he does the next best thing and *reconstructs* it out of "the few indications which permit its complete reconstitution, at least in its fundamentals."[20] His "faithful" reconstruction of ancient Greek algebraic laws, resulting in a full-fledged algebraic machinery for manipulating magnitudes of all kinds, both discrete and continuous, collapses the Greek distinctions between numbers and lines and enables him, among other things, to introduce, and fully operate with, fractions, unknown in Hellenistic mathematics, without any qualms whatever, all this being done "without recourse to the intervention of any foreign [i.e., non-Greek] stratagem"![21]

On the way to obtaining this mysterious night-creature, Greek Algebra, Marie formulates definitions, propositions and axioms (without clearly distinguishing between them), one such axiom implying[22] the knowledge of signed (±) magnitudes and numbers(!),[23] while Corollary IV introduces, without further ado, the exponential notation and the concept of power,[24] which is later supplemented, by means of a definition, by the concept and notation of the *n*th root.[25]

Having unearthed the Algebra *Speciosa* of the Greeks, Marie sums up his endeavor in the following words:

> Such is evidently the Algebra which the Greek geometers used tacitly when preparing the synthetic demonstrations of their theorems, after they had previously used analysis to discover them. It would be impossible to doubt that its principles were familiar to Archimedes and to Apollonius; but, as the Greek genius abhorred the invention of abbreviatory signs, the Greek geometers had these principles only in the form of translations into ordinary language, and as such non-transmissible; in this manner, even if they had wanted to make their logistical procedures visible, they could not have done it [?!].[26]

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19 *Ibid.*, p. 263.

20 *Ibid.*

21 *Ibid.*, p. 264.

22 This is Proposition VI.

23 "Proposition VI. *Axiome.*—Si une grandeur est composée de plusieurs autres, par voie d'additions et de soustractions, on peut changer à volonté l'ordre des additions et des soustractions, pourvu que chaque grandeur reste, dans tous les cas, additive ou soustractive" (p. 269).

24 *Ibid.*, p. 274.

25 *Ibid.*, pp. 278–81.

26 *Ibid.*, p. 281.

Because this statement, and generally Marie's approach, embodies in a nutshell assumptions that have been traditional in the historiography of Greek mathematics before and after Zeuthen, Zeuthen himself being, after all, only a much more sophisticated and insightful representative of the traditionally prevailing view, and precisely because of the straightforwardness, unaffectedness, and harshness of Marie's words, and because of his lack of historico-philosopical sophistication, it is indeed fitting to continue to unpack the traditional view through Marie's blunt and forthright assessment of Greek mathematics. So, here goes.

Not only were the propositions constituting his reconstruction of a Greek hidden algebra known to the Greek geometers, claims Marie, but they were indeed quite familiar to them, as they used them at every step, "not in the general form which we had to give to them, but, in the usual cases in which they dealt with two or three superimposed ratios, i.e., a ratio composed of two or three ratios."[27] A "proof" of this astonishing assertion is provided by Pappus's treatment of the locus of points having the ratio compounded of the ratios of their distances to *n* lines and their distances to *n* other lines equal to a given ratio, indifferent to the order of the compounded ratios! "We do not have the tract in which Apollonius treated this question, but it is abundantly clear that he must have known that the order we speak of was immaterial: the solution, whatever it might have been, must have shown him this; but if he had any doubt, Pappus would have told us."[28]

If the reader still has any doubts concerning this brand of historical reasoning, the concluding paragraphs of the chapter under discussion should clarify matters once and for all. Marie is aware that the demonstrations he has provided of the various algebraic laws allegedly used by Hellenistic mathematicians may not actually

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[27] *Ibid.*, p. 282. The reader who may be shocked by the "discovery" that Greek geometers calculated with double or triple superimposed fractions should also reflect on Marie's following ascription of knowledge to them: If $X = A \dfrac{B}{C\frac{E}{D}}$ then $A = X \dfrac{C\frac{E}{D}}{B} = X \dfrac{C}{B}\dfrac{E}{D}$, from which one gets, $X = A \dfrac{B}{C}\dfrac{D}{E}$. Apollonius, says Marie, "fait souvent usage de cette proposition; il la démontre de nouveau dans chaque occasion" (p. 277)!!!

[28] *Ibid.*, p. 282.

coincide with those genuinely used by the Greeks. It is possible, after all, that what the Greeks used were "intuitive demonstrations"![29] This, however, should not constitute any real objection to his stance, since he intended "to show only how, without changing their mental habits, they could have attained to the knowledge of these [algebraic] rules."[30] And he goes on:

> We also wanted to show the ease with which, without changing one iota in the methods of the Greeks, one could carry out, in a few minutes, with the greatest advantages, the revolution which has troubled the minds since Diophantus to Descartes, which has demanded so much effort, and the stages of which developed in such great obscurity.[31]

When the moderns quarrel about priority and the honor accruing to the inventor of the literal calculus of magnitudes, "which the public admires as an astonishing novelty," they should consider that "when Apollonius introduces the formula: *Rectam quae abscinditur ex diametro inter ordinatim applicatam et verticem sectionis*, these are actually letters, no more no less! We call this line *x*, it is much simpler, but, in reality, it is the same thing; there is nothing more in *x* than there is in *recta quae abscinditur*, etc., or, if there is, then it is the false idea that the translation of [algebraic] laws requires the representation of magnitudes as numbers." And this gem of a comparison between the geometry of the ancient Greeks and the algebra of the Renaissance ends appropriately in historiographic flurry:

> Magnitudes in Apollonius and Archimedes are straightly [longuement] literal, but they are literal, *i.e.*, arbitrary, and the reasoning with them is general, as are the spoken formulas; while, on the other hand, from Diophantus to Descartes, or to Viète, magnitudes will be represented by fixed numbers, the reasoning will lack any generality, and the results will not be anymore expressed by either spoken or written formulas, by means, or as a function, of the givens, but only by means of the resulting numbers, where there shall remain no trace of the operations by means of which these numbers were arrived at.
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> This is indeed, it seems to me, backward progress.[32]

It may not be progress (though if this, i.e., the dramatic changes mathematics underwent during the sixteenth and seventeenth centuries,

[29] *Ibid.*

[30] *Ibid.*

[31] *Ibid.*, pp. 282–83.

[32] *Ibid.*, p. 283.

is not progress, then we do not know what progress is), but it is certainly backwards! The least said about it the better, according to Toomer's principle of kindness in assessing R. Catesby Taliaferro's translation of Ptolemy's *Almagest*, "silence is the kindest comment one can make."[33] Those readers who have followed us attentively up to now know what is fundamentally wrong with Marie's methodology. All that needs to be added is that, but for its bluntness and outspokenness, Marie's position takes to their bitter end historiographic principles that have guided the mainstream in writing the history of mathematics until the seventies of the twentieth century. The basic methodological principle of the "mainstreamers" has been the separation of mathematical form from content coupled with its necessary byproduct, to the effect, that the passe-partout opening all ancient mathematical texts to our understanding is our present-day mathematical knowledge. The effects of this approach for the historical comprehension of ancient mathematical texts has been catastrophic.

This brings us chronologically to Zeuthen and his crucial contributions to the history of Greek mathematics, and the history of conic sections in particular. Since Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*, with all the due respect it deserves, has been our immediate historiographic *bête noire* throughout this book, which has in an obvious sense been written as an alternative to Zeuthen, little more needs to be said at this juncture, except to point out again that his work stands out by its qualities and insights (wrong as they are, by our account) in the ocean of the mainstream swimmers to which he himself belongs.[34] (In the remainder of this chapter, bringing our, necessarily scanty but representative, selection of historical works up to our own times, more will be said characterizing those works, having a direct bearing on Zeuthen's historiography of mathematics.)

It is not unfair to say that practically all historical studies of Greek mathematics written until 1975, say, share the same fundamental methodological assumption, namely that the proper access to an ancient mathematical text is through modern-contemporary mathematical insights, typically algebraic, which have the benefit of remov-

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[33] G. J. Toomer, *Ptolemy's Almagest* (New York, etc., 1984), p. vi.

[34] Heath's *Apollonius of Perga Treatise on Conic Sections* (Cambridge, 1896), which appropriates Zeuthen's *Die Lehre* lock, stock, and barrel, clearly belongs to the mainstream, as does his *A History of Greek Mathematics* (Oxford, 1921); both of them still deserve scrutiny.

ing the "trifles", accompanying as a rule ancient demonstrations, while disclosing the eternal truths they contain; and, intimately connected with this is the view that form and content are an inherently unstable, fickle and erratic twosome, living in sin, the cohabitation of which invites the unceremonious penetration of the mathematical historian, an ingress leading regularly to the couple's separation and to the assignation of new and lawful partners. No surprise, then, that, partaking in the same assumptions, these studies supply, more or less, equivalent, and often identical, interpretations of the Greek stuff. This being, then, the typical case, it is not really necessary to multiply examples beyond necessity. Those we already gave, as well as a few more, making their appearance in the sequel, should suffice to gain a faithful depiction of the real state of affairs.

Among the mathematically authoritative twentieth century histories of mathematics is Nicolas Bourbaki, *Eléments d'histoire des mathématiques* (Paris, 1969).[35] It consists of the systematic collection of the Historical Notes accompanying the collective author's *Eléments de mathématique.*[36] The characterization given above of traditional histories of mathematics fits Bourbaki's *Elements* up to the hilt. Indeed, to the historically sensitive reader, it seems often that the writers of the Historical Notes have transferred their (somewhat mythological)[37] structural approach and strict adherence to axiomatics and logical arrangement, as well as their fanatical insistence on absolute generality, from mathematics to history.

One of the original founders of the Bourbaki group was Jean Dieudonné, who, among many other things, wrote a mathematical text intended for the last three years of the French *lycée.*[38] In it he calls for a reform of French secondary mathematical education along the lines of the structural Bourbakist approach. What is utmost in his mind is the concern for unification of the various mathematical disciplines, artificially splintered and presented in splendid isolation, despite of their being actually one and the same discipline: "[T]he

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[35] Translated into English as *Elements of the History of Mathematics* (Berlin, etc., 1994); the French edition cited above is the second revised edition.

[36] More than 20 volumes of the *Eléments* have been published since its inception in the late thirties.

[37] See Leo Corry, *Modern Algebra and the Rise of Mathematical Structures* (Basel, etc., 1996), especially Chapter 7, "Nicolas Bourbaki: Theory of *Structures*," pp. 293–342.

[38] *Algèbre linéaire et géométrie élémentaire* (Paris, 1964), 3rd edition.

grotesque quarrels of 'synthetical' and 'analytical' geometers, as incomprehensible for us as the Byzantine discussions about the sex of angels, have filled up volumes during the nineteenth century. However, it is well known . . . that under this debris of another age hides itself always *one* and the same discipline, the *Linear Algebra* of modern mathematicians, which also comprises . . . the classical theory of systems of linear equations, and which has become . . . one of the most central [*sic*] and most efficacious theories of contemporary mathematics . . ."[39]

Dieudonné goes on: ". . . I believe that one is *always* [our emphasis] interested in trying *to comprehend* as well as possible what one does, and that there is a great formative virtue for the mind to search in its proceedings the parsimony of means and the narrow fit of hypotheses to conclusions, as far as this is possible." And a little bit earlier, on the same page: ". . . one of the advantages of Linear Algebra is that it permits to present all the developments of 'Elementary Geometry' in a perfectly rigorous fashion and *without effort*,"[40] against the considerable difficulty associated with the axiomatic treatment derived from the Euclidean presentation.

Now, all these, in themselves legitimate, insights, when it comes to the teaching of mathematics, become *mal à propos*, when it comes to the writing of the history of mathematics; and the feeling is inescapable that they did indeed serve as guiding methodological principles in the writing of the Historical Notes.[41] Thus, when in the Preface to the *Elements of the History of Mathematics*, Dieudonné (?) says that the reader will not find in it any anecdotal information and that "what has been attempted above all, for each theory, is to bring out as clearly as possible what were the guiding ideas, and how these ideas developed and reacted the ones on the others",[42] it seems clear, from reading the various Historical Notes making up the book, that the rhetorical form of ancient (and, in general, pre-modern) geometrical texts (and not only those) belongs to the anecdotal for

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[39] *Op. cit.*, pp. 11–12.

[40] *Ibid.*, p. 13.

[41] Corry, *op. cit.*, p. 335, says the following: "Bourbaki's historiography, as manifest in the *Eléments d'histoire* as well as in the individual writings of Dieudonné and of Weil, has been strongly connected with their overall conception of mathematics. In particular, they have applied similar criteria to differentiate important from unimportant ideas in both *present* mathematical research and *past* mathematical theories. Naturally enough, this has also been the case regarding the centrality of structures in mathematics."

[42] *Op. cit.*, p. V.

Dieudonné, and that the "guiding ideas" belong to the algebraic kernel of the various geometrical concepts discussed. It is this that enables him to overlook entirely the cognitive significance of notation and speak without any qualms of the solution of linear and quadratic equations by the Babylonians[43] and of "the product of two ratios of quantities" in Euclid's *Elements*.[44] Though he is aware that there are no explicitly stated rules of calculation in Babylonian texts, Dieudonné does not doubt that the Egyptians and Babylonians were "already in possession of a complete system of rules of calculations for whole numbers > 0, rational numbers > 0, lengths and areas."[45] Furthermore, when an ancient scribe "solves equations" of the same type, by following an identical procedure, in the absence of any explicit statement of a general rule, "the carrying out of calculations in the numerical cases treated makes the possession of such a rule more or less likely",[46] which in Dieudonné's practice means "certain". In short, for Dieudonné there seems to be no difference between the principles governing the writing of a historical text, like the *Elements of the History of Mathematics*, and the writing of a mathematical text, like *Algèbre linéaire*.[47]

Jean Dieudonné is a fine and faithful representative of Bourbaki historiography. Another is a towering figure of twentieth century mathematics, André Weil, who thinks that proposition IX.8 of the *Elements*, dealing with numbers in continued proportion, is actually a special case of the rule $a^{pq} = (a^p)^q$, for $q = 2$.[48] That this is so, and this is the only "argument" Weil gives, is easily recognized by anybody who has learnt at least school-algebra, and, better still, by any mathematician.[49] The problem is that Euclid had not learnt school-algebra, nor was he, strictly speaking, a "mathematician" *à la* Weil![50] For obvious reasons, then, he could not have recognized that the

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43 *Ibid.*, p. 47.

44 *Ibid.* Strictly speaking, this is not true.

45 *Ibid.*

46 *Ibid.*, n. 1.

47 This is how the theorem of Pythagoras appears in the latter, on page 87: *Si x, y sont deux vecteurs orthogonaux, on a* $||x+y||^2 = ||x||^2 + ||y||^2$. Dieudonné points out, on the preceding page, that the theorem is a particular case of $||\lambda x + \mu y||^2 = (\lambda x + \mu y | \lambda x + \mu y) = \lambda^2 || \times ||^2 + \mu^2 ||y||^2 + 2\lambda\mu(x|y)$.

48 "Who Betrayed Euclid?" in the *Archive for History of Exact Sciences*, vol. 19, no. 2 (1978), pp. 91–93, at 92.

49 *Ibid.*

50 The term *mathematician* is ultimately derived from the Greek μαθηματικός (meaning originally, *disposed to learn* (anything), and associated with the learning and

hidden subtext of the eighth proposition from the ninth book of his *Elements* was a special case of $a^{pq} = (a^p)^q$.

Proposition IX.8 is not for Euclid about the exponential notation, but rather about the properties of continued proportions. Not having been a modern mathematician, "the truth of the matter" for him, (the truth, the whole truth, and nothing but the truth) was wholly contained in the πρότασις of the proposition and not in any wildly anachronistic connection between its *hidden message* and the features of non-commutative groups,[51] laying in store for some 2300 years, to be unearthed in a future that must have been rather dim in Euclid's eye. Nobody in his right mind thinks that "the rules governing the use of the exponential notation are trivial,"[52] but not everybody is inclined, as Weil is, to trace them all the way back to Euclid, applying to the *Elements* misshapen and ahistorical interpretive standards.

Another French historian of mathematics, who is not a blind follower of the Bourbaki style, is Jean Itard. His *Essais d'histoire des mathématiques*[53] does not veer, however, from what has become standard fare in the history of Greek mathematics. Practically everything in the traditional assessment of the *Elements* is there: Book II lays the foundations of Geometrical Algebra; it deals with the solution of quadratic equations, a topic known to the Babylonians, who also solved some cubic and higher-degree equations; the fact that Al Khwarizmi proved his solution-recipes of quadratics by means of techniques drawn from Book II is somehow taken to point to a tradition of solving quadratics going back, at least, to Euclid; Book III establishes the notion of power of a point with respect to a circle; the selection criterion for the construction of regular polygons in Book IV was their constructibility by means of quadratic equations, etc., etc.[54]

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practicing of mathematics proper only by Aristotle) via the Latin *mathematicus*, and marked its first usage in English to designate an astrologer, astronomer, or mathematician, probably not before the XVIth century; as late as 1549, an English source could designate Ptolemy, Averroes, Aristotle, Galen, Hippocrates, and Cicero as "expert practicians in methamatic [*sic*] art" (*The Compact Edition of the Oxford English Dictionary*), s.v. *Mathematic.*

[51] Weil, *ibid.*

[52] *Ibid.*

[53] (Paris, 1984), collected and introduced by Roshdi Rashed.

[54] *Op. cit.*, pp. 91–92.

This brings us to what is, without doubt, the best and most important piece of historical scholarship on the *Conics* in the twentieth century, Toomer's commented edition and English translation of Books V–VII, extant only in Arabic,[55] and used extensively in our own book. The thorough study of Toomer's edition is a must for anybody wanting to reach an understanding of the *Conica* as a whole. Still, while an obvious improvement over Halley's Latin translation,[56] and more sensitive historically than Toomer's own article on Apollonius in the *Dictionary of Scientific Biography*,[57] this edition is, as we have mentioned in previous chapters, nevertheless, heavily biased in favor of modern symbolic notation (and, at times, too liberal, for our tastes, in its readiness for algebraic analysis), not only in the paragraphs "Mathematical Summary"[58] and "The Ancient Mathematical Background,"[59] but also in the body of the translation itself, where modern algebraic notation is used magnanimously throughout, and certainly in the critical Notes accompanying the text and translation.[60] It is clear that Zeuthen breathes down heavily Toomer's neck[61] and that,

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[55] G. J. Toomer, *Apollonius Conics Books V to VII The Arabic Translation of the Lost Greek Original in the Version of the Banū Mūsā*, Two Volumes (New York, etc., 1990).

[56] *Apollonii Pergaei Conicorum Libri Octo et Sereni Antissensis De Sectione Cilindri et Coni Libri Duo* (Oxford, 1710), two volumes.

[57] Ed. C. C. Gillispie, vol. I (1970), pp. 179–193; for instance, in the *D.S.B.* article, "geometrical algebra" is assessed wholly favorably (p. 184), while in his edition, Toomer considers the term "not altogether appropriate" (p. lxxxii). We, needless to say, welcome the change; Frank Manuel's statement, made in 1971, comes to mind and is certainly also applicable to Toomer's own two decades traversed between the two items of his scholarly bibliography: "Anyone who has maintained himself in an unswerving, upright posture over the past two decades has probably been dead on his feet" (Frank E. Manuel, *Freedom From History* (New York, 1971), p. x).

[58] *Op. cit.*, pp. xxviii–lxxxi.

[59] *Ibid.*, pp. lxxxi–lxxxv.

[60] Examples abound throughout the notes and commentaries as well as through the translation itself; it is enough to open the volumes at will to convince oneself of the accuracy of this remark. Consequently, we shall refrain from exemplary overkill, which is entirely superfluous for the sake of the present discussion. However, to expose the flavor of Toomer's heavily symbolic style of rendering Apollonius's rhetorical, geometrical exposition, we provide the following illustrations drawn from Book VII (pp. lxxii–lxxiii), representing Toomer's transcriptions of the corresponding Apollonian propositions:

VII 8 $D^2 : (d+\hat{d})^2 = (N\Gamma \cdot M\Sigma) : (M\Xi + \sqrt{(MN \cdot M\Xi)})^2$.

VII 9 $D^2 : (|d-\hat{d}|)^2 = (N\Gamma \cdot M\Sigma) : (|M\Xi - \sqrt{(MN \cdot M\Xi)}|)^2$.

VII 10 $D^2 : (d \cdot \hat{d}) = N\Gamma : \sqrt{(MN \cdot M\Xi)}$

[61] Toomer, like anybody approaching the *Conics* seriously, gives Zeuthen his rightly

despite Toomer's linguistic mastery and classical sensitivity, his original, primary, inchoate access to Apollonius is (as was Zeuthen's) through the mathematical, algebraic doorway.

This last statement needs clarification. How else should one enter an ancient mathematical text if not through the mathematical doorway? Is there any other accessible entrance? And if there is, what is it and why, if anything, is it preferable to the mathematical? An ancient text, mathematical, philosophical, literary, or whatever, is the product of a culture foreign to ours, whose concerns, values, aims, standards, ideals, etc., are, as a rule, as alien to ours as can be. Though, inescapably, all history is retrospective history, the approach described and decried by Detlef D. Spalt as *Resultatismus*,[62] that takes its bearings and criteria from what it sees as the *modern* outcome of a lengthy, linear, and necessary evolution of concepts and operations, looking always back to the past in light of its modern offspring, is necessarily a highly distorting approach, since it adopts unashamedly the perspective of the present to (in this order) judge and understand the past. Taken at face value, Percy W. Bridgman's statement that the past has meaning only in terms of the present is simply not true. The historian's stance is rather the opposite: the present has

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deserved praise. This he does both in his *D.S.B.* article and in his edition. Thus, in the former, he says fairly: "The fundamental modern work on Apollonius (and the ancient theory of conics in general) is . . . *Die Lehre* . . . It is indispensable for anyone who wishes to make a serious effort to understand the methods underlying the *Conics*" (p. 193). In the latter, the tribute paid to Zeuthen is obvious throughout; just one example should suffice: "For a deep analysis of the central topic of Book V . . . the reader should consult the admirable discussion of Zeuthen, . . . where it is treated from a modern viewpoint" (p. xxxviii). Neither here, nor anywhere else in his treatment, it seems, does Toomer have anything critical to say about Zeuthen's historiography.

62 Another name for the same blemished approach is "*Orwellsche 1984-Geschichtsschreibung für den grossen Bruder Vernunft*", in Detlef D. Spalt, *Vom Mythos der mathematischen Vernunft* (Darmstadt, 1981), p. 71. This book, unknown in the English-speaking world, is a formidable attack on the historiography of the Bourbaki school. See also Spalt's review of the two volumes edited by Dieudonné, *Abrégé d'histoire des mathématiques, 1700–1900* (Paris, 1978) in the *Frankfurter Allgemeine Zeitung*, issue of 3 December 1985, entitled, "Nach Bourbakis Fortschrittmodel", in which one can read the following: "Strukturalistische Mathematikgeschichtsforschung ist kaum etwa anderes als ein kriminalistisches Aufsuchen jener Begriffe (oder deren struktureller Pendants) in der Vergangenheit, die aus heutiger—sprich: Bourbakis—Sicht die 'wahre Natur' der mathematischen Theorien ausmachen. Lehrsätze oder Methoden früherer Zeit sind solcher Historie nun dann und insoweit von Bedeutung, als sie sich als 'Spezialfälle' oder 'Vorläufer' zeitgenössischer Verallgemeinerungen erfassen lassen."

meaning only in terms of the past. Prima facie and on principled grounds, therefore, an interpretation of any written, reasonably extensive, document belonging to an ancient culture, which results in its totally unproblematic and absolute assimilation to our own, is suspect.[63] That this is so is, more or less, acceptable when it comes to other cultural artifacts than mathematical ones, which seem to enjoy the privilege of perdurability and universality. The immunity from cultural specificity that mathematical truths command stems from the prevailing view that their outward appearance, their packaging, as it were, and their purely mathematical content, the packaged merchandise, as it were, are neutral, unrelated, and mutually independent items. It is a calamitous view and the root of all evil in the historiography of mathematics.

But there is another entrance into an ancient text, mathematical or not, one that does no violence to it, that does not break the inviolable unity of form and content and then enters victoriously through the shambles it created, with the claim that the text is now understood; it is rather an ingress that accepts willingly and respectfully

[63] In his last, posthumously published, book, *Conquest of Abundance* (Chicago and London, 1999), Paul Feyerabend makes a related point when speaking of the Presocratics and Parmenides in particular. He points out that the standard approach to the latter is not the only possible one, and gives an alternative, according to which "... we have not a premise, but a *beginning*..., not an argument but a *pattern of development*..., not conclusions but stages of development.... Even the logical part allows for a more historical interpretation than the one that is usually given" (p. 87). And he continues: "The existence of analogies of the kind just mentioned (which are cut off by a purely logical account) warns us not to be satisfied with reconstructions of ancient texts that rely on logic and mathematics alone. Such reconstructions may sound 'rational'—but they are irrational in the sense that they eliminate features which mobilized the judgment of contemporaries: their 'rationality' is for modern consumption only. Given a text we therefore cannot be content with a model that produces one part of it when another part is given and makes sense to a modern reader. The model must consider all elements ... and the textual as well as extratextual implications for the contemporaries" (p. 88). He clarifies what he means with the following example: "Examining archaic number concepts, many researchers assume that they share basic elements with later arithmetical notions and that they are numbers because they share these elements. Tables of addition, ... multiplication which are 'correct' when interpreted in modern terms seem to confirm the assumption. But such an interpretation cannot explain why counting people was ... supposed to endanger them or why it was ... assumed that counting a herd can diminish it. Nor can it explain why the oldest known number systems were dyadic and why this feature survived right into the beginning of arithmetic ... Numbers which have such features and such effects have to be different from the numbers examined by modern number theoreticians" (*ibid.*, n. 7).

the unity of the text, without pulverizing its inseparable aspects, bringing to its understanding both critical acumen and full acceptance of its outward appearance. This is the historical approach. The mathematical and the historical approaches are antagonistic. Whoever breaks and enters typically returns from his escapades with other spoils than the peaceful and courteous caller.

Let us be more specific. Faced with an ancient mathematical text, the modern interpreter has an initial choice. First is the mathematical approach. It consists of two steps: (1) try to find out how s/he would do it (solve the problem, prove the proposition, perform the construction, etc.) and then (2) attempt to understand the ancient procedure in light of his/her answer to step (1). Instead of this preeminently mathematical approach, however, the modern interpreter can refuse to decipher the text by appealing to modern methods, using for its understanding only ancient methods available to the text's author. This is the historical approach. Needless to repeat, the spoils of interpretation differ according to the two approaches followed.

The longstanding traditional approach has been the mathematical and we have encountered in this book numerous instances of it. Zeuthen is one such illustrious instance. Toomer is another. This is obvious when reading Apollonius and Toomer on Apollonius side by side, as his edition *cum* translation makes it easily possible.[64]

We can even find in the contemporary historiography of mathematics instances when the author identifies explicitly his line of attack as mathematical, as well as cases, when, in the course of his career, the historian switches lines of attack, from the mathematical to the historical; in the latter case, as is to be expected, the fruits of his historical labors differ dramatically in the two stages of his professional life.

Olaf Henrik Schmidt (1913–1996) is a Danish historian of mathematics whose access to ancient mathematical texts has always been through the mathematical[65] doorway. "He would start, say, from

[64] No additional examples are really needed.

[65] Qualifying "mathematical" by "modern" is superfluous, since the mathematical approach in question assumes that ancient mathematics is nothing but modern mathematics in disguise; as a matter of fact, the mathematics of all eras, ancient, medieval, Renaissance, modern, contemporary, is the same, mathematics pure and simple; the differences are trivial and external; some of the information on Olaf Schmidt's teaching was given to one of us orally by one of his former students, Dr. Lis Brack-Bernsen, at The Fifth International Conference on Ancient Mathematics, in Delphi, 17–20 August 2000.

Hilbert's geometry, theory of congruence and parallels, and then point out how much of this is to be found in Book I of the *Elements*. After praising Euclid for getting that far, he would demonstrate how little was missing, making 'repairs', so to speak."[66] "Or, he would look at the axioms of magnitudes and demonstrate that Axiom 3 can be proved from Axiom 2, if another obvious axiom is supplied . . . Everything was seen from the modern point of view of consistence and completeness. Or he would show how little was to be supplied to conceive Book V [of the *Elements*] as a perfect introduction of Real Numbers."[67]

That Schmidt's historical methodology is mathematical is amply clear from the reports above. But we also have Schmidt's own words to vouchsafe the reports of his pupils. In his reconstruction of Archimedes' theory of equilibrium and centers of gravity, "A System of Axioms for the Archimedean Theory of Equilibrium and Centre of Gravity,"[68] considered by himself as his best work,[69] Schmidt characterizes his reconstruction as being "as close as possible to the *Archimedean way of thinking*," but deviating "quite a bit from the *Archimedean way of writing*",[70] failing to discern that preserving the former, while deviating considerably from the latter is an impossible exercise in intellectual equilibristics. He goes on, revealing his true mettle: "To obtain greater clarity in the exposition I shall make use of simple concepts from modern mathematics, such as e.g. a mapping of a set into a set, a product of two sets, or a relation in a set . . . . any historian of mathematics when dealing with ancient mathematics uses a notation different from the original one, which is often non-existant [*sic*]".[71] Explaining his procedure, which he deems as close as possible to Archimedes' way, Schmidt picks and chooses his axioms from amongst those used by Archimedes, abandoning some and adding new ones, non-existent in the entire Archimedean corpus![72]

Discussing the center of gravity of plane polygons, Schmidt formulates and proves a lemma,[73] in which the conclusion is written:

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66 Written communication from C. M. Taisbak to S. Unguru, Sept. 3, 2000.

67 *Ibid.*

68 *Centaurus*, vol. XIX (1975), pp. 1–35.

69 Taisbak's communication (n. 66).

70 *Op. cit.*, p. 3.

71 *Ibid.*

72 *Ibid.*, pp. 4, 17.

73 *Ibid.*, pp. 19–23.

$G_1^{(n)}G_2^{(n)} \rightarrow \infty$ as $n \rightarrow \infty$; then, he says: "Archimedes does essentially the same thing but he expresses himself, except for notation [!]" somewhat differently . . .[74] His Final Remarks deserve quotation *in extenso*, since they exude mathematical innocence and contain in a nutshell the sublimation of all that is wrong with the mathematical approach to ancient texts:

> The first reaction of a reader of this paper is most likely somewhat negative.This may be true . . . for a reader with, a background in Greek classics rather than in mathematics. He may maintain that he does not recognize any of Archimedes' thoughts in this paper [this is true!], and he may doubt that an investigation like the one carried out here serves any purpose whatsoever [this is also true!]. To this I want to answer that I firmly believe that if the present paper were shown to Archimedes, his reaction would not be entirely negative. He might even nod approvingly, and admit that he actually wanted to do something of the kind done here, but that lack of the proper mathematical tools prevented him from expressing his ideas as abstractly as we have done here.[75]

And:

> . . . what is attempted in this paper is to extend the Archimedean way of reasoning [Indeed!]. . . . When reading an ancient piece of mathematics one often has the benefit [or, rather, the curse] of a knowledge of a better and more extensive exposition of the same subject and thus the possibility of evaluating the ancient contribution.[76]

This is certainly a fair self-characterization of Schmidt's paper.

Christian Marinus Taisbak was Schmidt's Ph.D. student. His published dissertation[77] embodies the historiographic insights of his master. Thus, starting with the modern concept of equivalence classes and what he calls the fundamental theorem of arithmetic,[78] Taisbak manages to show how the theory of ratios and proportions appearing in the *Elements* is an instantiation of the modern theory of equivalence

[74] *Ibid.*, p. 22.

[75] *Op. cit.*, pp. 34–35.

[76] *Ibid.*, p. 35.

[77] *Division and Logos A Theory of Equivalent Couples and Sets of Integers Propounded by* Euclid *in The Arithmetical Books of the Elements* (Odense, 1971).

[78] The theorem of Gauss: If a number divides a product of two factors and is prime to one of them, it divides the other.

classes.[79] His book is both highly ingenious and heavily anachronistic. It should be read with great care and always with an eye on the Euclidean text, which, as a rule, Taisbak quotes in the original. Some examples are in order. Pointing out the coherence of the arithmetical books of the *Elements* and their character as a first in number theory, Taisbak says: "... it must be possible to understand the contents of those books by reading the text literally. Two prerequisites are needed to the task: You must know Greek;[80] for much information is hidden in the peculiar Greek terminology. And you must know what number theory is about, (and do not hesitate to use some very modern account, since your knowledge of Greek will prevent you from reading modern concepts into the text which are not there to the Greek mind.)[81] The proof of the pudding being in the eating, we must say that the concoction "Greek *cum* Mathematics" proved itself totally ineffective for the task at hand, plainly quack medicine against the ills it was meant to prevent; it turns out that one of the two ingredients in the elixir simply overwhelms the other. "The general aspect of this book is, accordingly, not historical, but mathematical—and it is bound to be a defence of an unitarian view of Euclid."[82] This is true.

There are attempts at "saving" Euclid,[83] a whole "Theory of Lambda Classes,"[84] the extension of the concept of *logos*[85] "from couples to ordered sets [!] of numbers," in which "Euclid diverges decisively from any isomorphism with fractions",[86] and in connection

[79] From equivalent couples, which he uses to deal with Book VII, Taisbak passes on to equivalent sets in dealing with Books VIII and IX (*op. cit.*, pp. 84–110).

[80] Taisbak was trained as a classicist and is an outstanding specialist in the field.

[81] *Op. cit.*, pp. 12–13.

[82] *Ibid.*, p. 13.

[83] *Ibid.*, p. 31.

[84] *Ibid.*, p. 72, *passim.* The relation λ, by the way, means "has the same logos as." Some nuggets: "A *definition* of a *minimi* [sic] in a *Lambda Class* is tacitly understood. The *existence* of such a couple in any Lambda Class is considered self-evident; we know that it is the cosequence of the Principle of Well-Ordering (*W*): the specific property of positive integers that any non-empty subset of positive integers has a least element. The *uniqueness* of the minimi is a consequence of the ordering of the integers" (p. 78). God alone knows what all this has to do with the *Elements*!

[85] About which it is correctly stated that its theory "*is not a theory of fractions*" (*ibid.*, p. 84).

[86] *Ibid.*

with which it is said: "In the set of ordered sets of $n + 1$ numbers the relation λ determines a partition of the set into equivalence classes, which I shall denote as 'Lambda Classes of the n'th order'. . . . The whole of the 8'th book of the Elements is devoted to the examination of these classes . . .,"[87] and other insights *ejusdem farinae.* It should be clear, then, that *Division and Logos* is a book whose insights are primarily mathematical, not historical. To remove any possibly lingering doubt, Taisbak ends the ninth chapter of his book, dealing with Equivalence Classes, with the following resoundingly eloquent and convincing words, the last of which involve a reverberating *contradictio in adjectu*:

> If it is true that the essentials of Euclid's theory of numbers aim at giving a deductive foundation to these arithmetician's considerations about single and double tables [of multiplication] . . . then there is no reason why classes formed by *three or more* tables should not be made the subject of a similar theory.
>
> Euclid's algebra [?!] does comprise such a theory, which may be brought to light by an interpretation of the 8'th and 9'th books of Elements. I shall try to expose that theory, paying the greatest possible regard to Euclid's own terminology [!?][88]

"Equivalence Classes and respect for Euclid's own terminology" is what we call in Hebrew אפכא מסתברא, the opposite of what is intended holds; it is an impossible mixture, a logical oxymoron, and that is the least one can say about it.

But Taisbak is also the author of the best interpretation of Book X of the *Elements*.[89] "There is no doubt that this is a wonderfully witty, premeditatedly nonpolemical, understated, civilized, accommodating, and important little book, whose author knows well both his Greek and his mathematics and whose feet are firmly anchored in the historical soil."[90]

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[87] *Ibid.*, p. 86. Here is one of the three propaedeutic definitions for vindicating the claim that Books VIII and IX are about Lambda Classes: "The Lambda Class $L_n = \{(a_0, a_1, a_2, \ldots, a_n)\}$ is the set of ordered sets of $n + 1$ numbers $(b_0, b_1, b_2, \ldots, b_n)$ of which it holds that $(b_0, b_1, \ldots, b_n)\ \lambda\ (a_0, a_1, \ldots, a_n)$. L_n is called a Lambda Class of the n'th order" (*ibid.*).

[88] *Ibid.*, pp. 82–83.

[89] *Coloured Quadrangles A Guide to the Tenth Book of Euclid's Elements* (Copenhagen, 1982).

[90] Essay review of Taisbak's book, by S. Unguru, in *Historia Mathematica*, vol. 14 (1986), pp. 67–70, at p. 68. The conclusion of the review is: "[T]his is a marvelous, sensitive, tasteful, polished, subdued, and significant little volume, written by

There is a world of difference between *Division and Logos* and *Coloured Quadrangles*; the former had approached its topic through the mathematical door, while the latter's approach is historical. It can, therefore, serve as a *specific* illustration of the historical mode of writing the history of Greek mathematics. Basically, Taisbak restates Euclid's Book X in the language of "colors," which is a neutral language with respect to the nature of Greek mathematics, and which leaves the original features of the book untouched.

Taisbak's "color vocabulary" is conceived precisely, astutely, and insightfully as a means of adhering intimately to the Greek terms ῥητός, μέσος, and ἀλογός, performing, as it were, a *sui generis* "translation" of Euclid's geometric discourse, a mere labeling, so to speak, according to which the above terms are designated respectively as *red, amber,* and *obscure.* By means of these designations, Taisbak manages to explain sympathetically Euclid's way of thinking without thereby violating the geometric tenor of Book X. The result is a true guide to the book "for those who may have lost their faith in the commentaries by . . . Heath . . . that they may partake in the growing debate about the nature of Greek . . . mathematics."[91] Taisbak's final conclusion, and it is a deeply satisfying historical conclusion, is that "the subject of the X'th book is a very limited one, that the reasonings are simple and straightforward manipulations with quadrangles in the idiom of commensurability, and that any association with theories of equations is *ex post facto* modernism."[92]

Taisbak's analysis shows convincingly that Book X is a book of Greek *geometry,* to which both measuring and the concept of real number are foreign, and that all the traditional arithmetico-algebraic interpretations of it distort its true nature. Taisbak's interpretation is refreshingly historical. Not so, his anachronistic reading of the arithmetical books of the *Elements* in *Division and Logos.* Despite the enlightened attempt to wed ancient Greek to modern mathematics in it, the offspring of this *mésalliance* turned out bastardly inappropriate and dastardly offensive, a result to be expected, after all, from a marriage made in hell, while the single motherhood, *Historia,* under whose

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a man who has something important to say and whose contribution should, in a better world, put to rest the ahistorical arithmetico-algebraic interpretations of Book X" (*ibid.*, p. 69).

[91] *Coloured Quadrangles*, p. 77.

[92] *Ibid.*

aegis *Coloured Quadrangles* was conceived, has produced a superbly satisfying study.

It ensues that, in practice, no compromise is possible between the mathematical and historical methodological principles.[93] Adopting one or the other has fateful consequences for one's research, effectively determining the nature of the results reached and the tenor of the inferences used in reaching them. We believe the historical approach to be the fitting approach for the study of ancient mathematical texts. It has guided our endeavor in writing this book. It is for the reader to judge the results.

[93] The same is true, *mutatis mutandis*, with respect to the study of literary, or any other kind of ancient, texts. An apt example is provided by A. E. Housman's outburst, in his Cambridge Inaugural Lecture of 1911, against the application of the term "exquisite" to a passage from Horace: "Exquisite to whom? Consider the mutations of opinion, the reversals of literary judgment, which this one small island has witnessed in the last 150 years: what is the likelihood that your notions or your contemporaries' notions of the exquisite are those of a foreigner who wrote for foreigners two millenniums ago? And for what foreigners? For the Romans, for men whose religion you disbelieve, whose chief institution you abominate, whose manners you do not like to talk about, but whose literary tastes, you flatter yourself, were identical with yours. . . . Our first task is to get rid of [our tastes], and to acquire, if we can, by humility and self-repression, the tastes of the classics" (quoted from Daniel Mendelsohn, "The Tale of the Two Housmans", *New York Review of Books*, vol. XLVII, Nr. 13, August 10, 2000, pp. 58–65, at p. 64).

APPENDIX

AN ENGLISH TRANSLATION OF BOOK IV OF APOLLONIUS'S *CONICA*[1]

Translator's Preface

Until now, the English reader of the *Conica* has had access to Books I–III and to Books V–VII. Book VIII, of course, is lost, and there is little hope readers of *any* language will ever know exactly what it contained. The Greek text of Book IV, however, has had no more difficulty surviving than the Greek texts of Books I–III. Yet, although Heath, in his *Apollonius of Perga: Treatise on Conic Sections*, produced an English translation of the opening letter and certain of the enunciations of the propositions in Book IV, the translation below is, to my knowledge, the *first* English translation ever carried out of the *whole* book. Why this should be the case, no doubt, has partly to do with the fact that until recent years the history of mathematics has been approached almost exclusively through the "mathematical door," as we discussed in the last chapter. Approached through *this door*, Book IV is simply not as interesting as Books I to III. Indeed, if one sees the *Conica* only as an *algebraic* text disguised in geometric language, where the essence of the conic sections is summarized by their *symptomata* understood as *equations*, then Book IV must truly seem pedantic and repetitious and, in short, as Heath says, dull. However, if our chapter on Book IV above has been at all persuasive, then it should be plain that the endless cases of intersecting conics covered in Book IV, in truth, reveal fundamental concerns in Apollonius's essentially *geometric* approach to conic sections. So, far

[1] The translation presented here has been previously published by the Green Lion Press as a companion volume to their edition of Taliaferro's translation of Books I to III. While the Green Lion version has none of the critical apparatus obligatory in a scholarly text, *i.e.* bracketed words and phrases, which can be an annoyance for the casual reader, it does contain most of the notes on the text and also an introductory essay. This appendix, on the other hand, contains the full-fledged scholarly edition of my translation. We are grateful to the Green Lion Press for allowing us to present this complete version of the translation here.

from being dull, Book IV can provide real insight into the way conic sections were understood by Apollonius.

Any translation, naturally, involves compromise, and this one is no exception. On the one hand, in keeping with what has been said in this book, I have tried to keep out all foreign and anachronistic elements as much as possible. In particular, there is no modern symbolism in the translation, no matter how seemingly benign. I have also used, with one exception, different words for the different ways in which conic sections are described as meeting. Thus, I translate συμβάλλειν as "to meet", τέμνω as "to cut" and συμπίπτειν as "to intersect". The exception to this is my translating both ἐφάπτεσθαι and ἐπιψαύειν as "to touch". On the other hand, I have used throughout Latin instead of Greek letters. However, I have employed a single lettering scheme in which (with two exceptions—"theta" and "psi") the Latin letters correspond to the Greek either in sound or shape, for example, "G" corresponds to "gamma" and "H" to "eta". (The complete lettering scheme follows this preface.) A more serious deviation from the Greek has been the use of the simple passive imperative for the perfect passive imperative. This has not been done without serious reservations, and I discuss these in note 9 below.

As for the Greek text, I have used Heiberg's edition throughout. Heiberg's Latin translation, despite its free use of algebraic notation, has also been of great use, as has been Ver Eecke's French translation.

Michael N. Fried

Lettering Scheme for Diagrams in Book IV

α	A
β	B
γ	G
δ	D
ε	E
ζ	Z
η	H
θ	Q
ι	I
κ	K
λ	L
μ	M
ν	N
ξ	C
ο	O
π	P
ρ	R
σ	S
τ	T
υ	U
φ	F
χ	X
ψ	V
ω	W

APOLLONIUS'S *CONICA*: BOOK FOUR

Apollonius to Attalus,[1] *greeting!*

Earlier, I presented the first three books of my eight book treatise on conics to Eudemus of Pergamum, but with his having passed away I decided to write out the remaining books for you, because of your earnest desire to have them. To start, then, I'm sending you the fourth book. This [book] treats of the greatest number of points at which sections of a cone can meet one another or meet a circumference of a circle, assuming that these do not completely coincide, and, moreover, the greatest number of points at which a section of a cone or a circumference of a circle can meet the opposite sections. Besides these [questions], there are more than a few others of a similar character. Conon of Samos presented the first mentioned [question][2] to Thrasydaeus without giving a correct demonstration, for which he was rightly attacked by Nicoteles of Cyrene.[3] As for

[1] Nothing definite, to my knowledge, is known about Attalus. Toomer, however, argues convincingly that this Attalus cannot be King Attalus I of Pergamum, who reigned in Pergamum at the time Apollonius possibly was writing the *Conics*, for, if King Attalus were intended, surely Apollonius would have referred to him using the word "King" (βασιλεύς) which he does not (see Toomer, "Apollonius of Perga," p. 179). Toomer elsewhere has "speculated whether Apollonius' addressee is to be identified with the mathematician Attalus of Rhodes (known mainly from Hipparchus' criticism in his commentary on Aratus), but apart from chronological difficulties, Ἄτταλος is too common a name for any certainty" (*Apollonius Conics Books V to VII*, p. xii, n. 2).

[2] Following Heath (*Apollonius of Perga*, pp. lxxii–lxxiii), I take the first question to be that concerning the greatest number of points at which sections of a cone can meet one another or meet a circumference of a circle, the second, the greatest number of points at which a section of a cone or a circumference of a circle can meet the opposite sections, and the third, the greatest number of points at which the opposite sections can meet one another, while the other questions "of a similar character," I take to be those concerning the number of points conic sections, circles, and opposite sections can touch one another. This seems to me a reasonable division and one that is completely consistent with the structure of the book.

[3] Of the three names mentioned here, Conon of Samos, Thrasydaeus, and Nicoteles of Cyrene, only about Conon can we say something definite. We know that he communicated with Archimedes and was greatly respected by him. We also know that he had died before Archimedes published his *Quadrature of a Parabola*, as Archimedes tells us in the letter to Dositheus opening that book. This means that

the second [question], Nicoteles, in replying to Conon, only mentions that it can be demonstrated, but I have found no demonstration either by him or by anyone else. Regarding the third and similar [questions], however, I have not found them even noticed by anyone. And all these things just spoken of, whose [demonstrations] I have not found anywhere, require many and various striking theorems, of which most happen to be presented in the first three books [of my treatise on conics], and the rest in this book. The investigation [of these theorems] is also of considerable use in the synthesis of problems and limits of possibility (διορισμούς). So, Nicoteles was not speaking truly when, for the sake of his argument with Conon, he said that none of the things discovered by Conon were of any use for limits of possibility; but even if the limits of possibility are able to be obtained completely without these things, yet, surely, some matters are more readily perceived by means of them, for example, whether [a problem] might be done in many ways, and in how many ways, or again, whether it might not be done at all.[4] Moreover, this preliminary knowledge brings with it a solid starting point for investigations, and the theorems are useful for the analysis of limits of possibility.[5] But apart from such usefulness, [these things] are also worthy of acceptance for the demonstrations themselves: indeed, we accept many things in mathematics for this and no other reason.[6]

Conon died about the time Apollonius was born, ca. 240 B.C.E., and, therefore, the questions with which Apollonius was engaged in Book IV were already being asked a generation before Apollonius. About Nicoteles, Knorr tentatively mentions the possibility that Nicoteles is none other than Nicomedes. He says, ". . . Apollonius names one Nicoteles of Cyrene who was especially harsh in his criticisms of Conon's studies of intersecting conics. One is tempted to suggest that a scribal slip in the transmission of the Conics has altered an original 'Nicomedes'. Clearly, in both contexts an abrasive individual is encountered, engaged in advanced geometrical studies associated with the work of geometers around the middle of the 3rd century or a bit later. But beyond this, there seems to be no other evidence to argue for the identification" (*The Ancient Tradition of Geometric Problems*, p. 282, n. 41).

[4] Literally, ". . . whether [something] might come to be (ἂν γένοιτο) in many ways, and in how many ways, or again, that it may not come to be." I am following Heiberg here in taking the 'something' to be 'a problem'. Heiberg's translation reads ". . . velut problema compluribus modis vel tot modis effici posse aut rursus non posse . . ." Ver Eecke similarly translates the phrase as referring to problems and their solutions: ". . . soit la possibilité de solutions multiples, ou en nombre déterminé, soit, au contraire, l'impossibilité d'une solution."

[5] P. ver Eecke translates the phrase πρὸς τὰς ἀναλύσεις δὲ τῶν διορισμῶν simply as ". . . aux développements des discussions." I have chosen, however, to emphasize a more 'technical' reading of διορισμος, namely, as 'limits of possibility'.

[6] Apollonius stresses the intrinsic interest of his subject also in the letter to Attalus

Proposition 1

If a point be taken outside a section of a cone or circumference of a circle, and from this same [point] two lines are extended towards the section, of which one touches [the section] and the other cuts [the section] at two points, and [if] the line cut off inside [the curve] be divided in that ratio which the whole [line] cut off has to the part outside bounded between the point and the curve,[7] so that the homologous lines are at the same point,[8] [then] the line drawn from the point of contact to the [point of] division [*i.e.* the point dividing the inside segment] will intersect the curve, and the line drawn from the point of intersection to the outside point will touch the curve.

For let *ABG* be a conic section or circumference of a circle [see fig. 1], and let the point *D* be taken outside [the section];[9] from [*D*]

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that introduces the fifth book. He writes, "As for the propositions in which we speak of the minimum lines, we have separated them out and treated them individually, after much investigation, and have attached the discussion of them to the discussion of the maximum lines which we mentioned above, because of our opinion that students of this science need them for the knowledge of analysis and determination of problems and their synthesis, *not to speak of the fact that they are one of the subjects which deserve investigation in their own right* [emphasis added]," (Toomer's translation from the Arabic in *Apollonius* Conics *Books V to VII*, pp. 3–4).

[7] I am translating γραμμή as "curve" to avoid any confusion with the other *straight* lines (εὐθεῖαι) spoken of in the proposition.

[8] What Apollonius is saying here is that the four lines involved in the proportion are laid out *in the same direction*, thus, referring to the diagram (fig. 1), *GD* and *GZ*, which correspond in the proportion, must both extend from the same point *G*.

[9] Strictly speaking this should be translated "let some point D have been taken (εἰλήφθω) outside." εἰλήφθω is the *perfect passive imperative* of λαμβάνειν. In almost all Greek mathematics this is the form used in that part of a demonstration where constructions are prescribed (cf. Heath, *Euclid*, I,242). I have translated it as an *ordinary* passive only to avoid an overly awkward English translation, especially where there is a string of such commands. However, this is not done lightly since the ubiquitous use of this grammatical form may, indeed, point to a significant aspect of Greek mathematical demonstrations. On this subject, Lachterman writes, With only a few notable exceptions, Euclid chooses to put these verbs [verbs for "operation"] in the perfect passive imperative. Bisecting a line-segment at a point is expressed as "let it have been cut in two" (*tetmestho . . . dicha*) . . . The importance of this stylistic trait is twofold: First, Euclid does not give instructions or permission to a reader to carry out a specified operation but casts the operation into impersonal, passive form; second, the perfect tense tells us that the relevant operation has already been executed prior to the reader's encounter with the unfolding proof . . . Now we are indeed in unfamiliar territory. As though to strengthen the hand of the Speusippeans, Euclid invites us, not to perform an operation on our own, nor to observe him in performing the operation before our eyes, but rather to consider the operation as already anonymously performed before the 'present moment' in which we are following and taking stock of the movement from enunciation to conclusion. This verbal operator does not so much suppress time as shift

let *DB* touch [the section] at point *B* and let *DEG* cut the section at [points] *E,G*, and let *GZ* have to *ZE* the ratio that *GD* has to *DE*.

I say that [the line] drawn from *B* to *Z* will intersect the section, and [the line drawn] from the [point of] intersection to *D* will touch the section.[10]

For let *DA* be drawn from *D* touching the section, and let *BA* be joined cutting *EG*, if possible, not at *Z*, but at *H*. Now, since *BD,DA* touch [the section], *BA* [is drawn from] the points of contact, and *GD* goes through [*AB*], cutting the section at *G,E* and [intersecting] *AB* at *H*, as *GD* is to *DE*, so is *GH* to *HE* [III.37][11] But this is absurd, for it was assumed that as *GD* is to *DE* so is *GZ* to *ZE*.[12]

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it backwards, into an unnoticed past during which neither teacher nor student was necessarily on hand" (*The Ethics of Geometry*, p. 65). Taisbak beautifully puts the case as follows, "It might be appropriate at this stage to introduce that effective constructor, The Helping Hand, the well-known factotum in Greek geometry, who sees that lines be drawn, points be taken, perpendiculars dropped, etc. The perfect imperative passive is its verbal mask. No one who has read Euclid's *Elements* in Greek will have missed it. Never is there any of the commands or exhortations so familiar from our own classrooms: 'Draw the median from vertex A', or 'Cut the circle by that secant', or 'Let us add those squares together'. The Helping Hand was always there to see that these things were done. I wonder how Europe ever inherited Greek mathematics without the perfect imperative passive" ("*Elements of Euclid's* Data," p. 144).

[10] Following this, there is an odd remark, which I have omitted from the translation. Heiberg takes the remark to be an interpolation and also omitted it from his Latin translation. The text reads: "For indeed since *DG* cuts the section in two points, it is not a diameter. It is possible, therefore, to draw a diameter through *D*, and, thus, also a tangent [through *D*]." One can only guess that the interpolator thought there was some need to justify Apollonius's next step, namely, the drawing of a tangent from *D* to the section. Yet, good intentions aside, the remark is worse than irrelevant; it is false. Not only can one make the obvious objection that if the section is an ellipse, *DG* can cut the section in two points and still be a diameter, but also one can object that if *D* is in the angle contained by the extension of the asymptotes beyond the center, then one can draw a diameter through *D* but, as Apollonius specifically points out in *Conics* II.49 (further supporting the claim that this is the work of an interpolator), one cannot draw a tangent to the section through *D*.

[11] *Conics* III.37: "If two straight lines touching a section of a cone or circumference of a circle or opposite sections meet, and a straight line is joined to their points of contact, and from the point of meeting of the tangents some straight line is drawn across cutting the line (of the section) at two points, then as the whole straight line is to the straight line cut off outside, so will the segments produced by the straight line joining the points of contact be to each other."

Throughout the text, I shall indicate propositions in the *Conics* only by book and proposition number. Thus, II.49 refers to *Conics*, Book II, and proposition 49.

[12] The problem is that then *GZ:ZE::GH:HE*, so that two *different* points would divide the segment *GE* in the *same* ratio.

Therefore, *BA* does not cut *GE* in a different point [from *Z*]; therefore, it cuts *GE* at *Z*.

Proposition 2

The things [above] were demonstrated for all the sections together (κοινῶς). However, regarding the hyperbola only, if *DB* touches [the hyperbola] while *DG* cuts [it] at two points *E,G*, [and if] the point of contact, *B*, is contained by *E,G*, and *D* is inside the angle contained by the asymptotes, then the proof is carried out similarly [to that in proposition 1]. For from *D* it is possible to draw another line, *DA*, touching [the hyperbola] and the rest of the proof is done similarly.

Proposition 3

The same things being the case,[13] let the points *E,G* [fig. 2] not contain the point of contact, *B*, between them, [and let] *D* be inside the angle contained by the asymptotes. Therefore, from *D*, it is possible to draw another line, *DA*, touching to the section, and the rest is proven as before.

Proposition 4

The same things being the case, if the points of intersection[14] *E,G* [fig. 3] contain the point of contact *B*, while the point *D* be in the angle adjacent to that contained by the asymptotes, [then] the line from the [point of] contact to the point of division intersects the opposite section, and the line drawn from the point of intersection [to point *D*] will touch the opposite section.

Let *B* and *Q* be opposite [sections], let *KL* and *MCN* be asymptotes, and let point *D* be in the angle *LCN*. Furthermore, let *DB* [drawn] from *D* touch, while *DG* cut [one of the sections], let the points of intersection *E,G* contain the point of contact *B*, and let *GZ* have to *ZE* the same ratio that *GD* has to *DE*.

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[13] Namely, *ABG* is a hyperbola, point *D* is taken outside the hyperbola, *DB* is drawn touching the hyperbola at point *B,DEG* is drawn cutting the hyperbola at *E* and *G*, and *EG* is divided at *Z* such that *GZ:ZE::GD:DE.*

[14] The points of intersection, that is, of *DG* and the hyperbola, *ABG*.

It is to be shown that [the line] joined from *B* to *Z* will intersect the section *Q*, and that [the line] from the point of intersection to *D* will touch the section.

Let *DQ* be drawn from *D* touching the section, and let the line joining *QB* fall [on *DG*], if possible, not on *Z*, but on *H*. Therefore, as *GD* is to *DE*,*GH* is to *HE* [III.37]. But this is absurd, for it was assumed that as *GD* is to *DE*,*GZ* is to *ZE*.

Proposition 5

The same things being the case,[15] if the point *D* be on an asymptote [fig. 4], the line drawn from *B* to *Z* will be parallel to that asymptote.

For let these same things be supposed, let point *D* be on one of the asymptotes *MN*. It is to be shown that the line drawn from *B* parallel to *MN* will fall on the point *Z*.[16]

For if not, let the line be *BH*, if possible. But then as *GD* is to *DE*,*GH* will be to *HE* [III.35],[17] which is impossible.

Proposition 6

If a point be taken outside a hyperbola, and from this [point] two lines are drawn to the section, one of which touches [the section] while the other is parallel to one of the asymptotes, and [if the part of the latter line] inside the section be set equal to the [part] cut off between the section and the point, [then] the line joined from the point of contact [of the former line] to the produced point[18] will

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[15] Namely, that *B* and *Q* are opposite sections, *KL* and *MCN* are the asymptotes, *DB* touches the section at *B*,*DG* cuts the section at *E* and *G*, and *GZ*:*ZE*::*GD*:*DE*.

[16] Showing that the line *BZ* will be parallel to the asymptote and showing that the line through *B* parallel to the asymptote passes through *Z* are the same provided one assumes, what is equivalent to *Elem.*,I.post.5, that only one line can be drawn through a given point parallel to a given line. Evidently, Apollonius takes this to be so obvious that he sees no need to point it out.

[17] *Conics* III.35: "With the same things being so [namely, if some point is taken on one of the asymptotes of an hyperbola, and a straight line from it touches the section, and through the point of contact a parallel to the asymptote is drawn], if from the point taken some straight line is drawn cutting the section at two points, then as the whole straight line is to the straight line cut off outside, so will the segments of the straight line cut off inside [by the line drawn parallel to the asymptote] be to each other."

[18] That is, the endpoint of the line inside the section that has been set equal to the line cut off between the outside point and the section.

intersect the section, and [the line] drawn from the point of intersection to the point outside will touch the section.

Let *AEB* be the hyperbola and let *D* be some point taken outside [see fig. 5], and, to start, let *D* be inside the angle contained by the asymptotes, and from [*D*] let *BD* touch [the hyperbola], let *DEZ* be parallel to one or the other of the asymptotes, and set *EZ* equal to *DE*. I say that [the line] joined from *B* to *Z* will intersect the section, and [the line] from the point of intersection to *D* will touch the section.

For let *DA* be drawn touching the section, and join *BA* cutting *DE*, if possible, not at *Z* but at some other [point] *H*. Then *DE* will be equal to *EH* [III.30].[19] But this is absurd, for it was assumed that *DE* is equal to *EZ*.

Proposition 7

The same things being the case, let *D* be in the angle adjacent to that contained by the asymptotes [see fig. 6]. I say that the same things will come to pass.

For let *DQ* be drawn touching the hyperbola, and join *QB* falling [on *DE*], if possible, not at *Z*, but at *H*. Therefore *DE* is equal to *EH*. But this is absurd, for it was assumed that *DE* equals *EZ*.

Proposition 8

The same things being the case, let *D* be on one of the asymptotes [fig. 7], and let the remaining [constructions] be the same.[20]

I say that [the line] drawn from the point of contact to the end of the line cut off[21] will be parallel to the asymptote on which *D* is [situated].

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[19] *Conics* III.30: "If two straight lines touching an hyperbola meet, and through the points of contact a straight line is produced, and through the point of meeting a straight line is drawn parallel to some one of the asymptotes and cutting both the section and the straight line joining the points of contact, then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section."

[20] I am following ver Eecke here who translates καὶ τὸ λοιπὰ γινέσθω as ". . . et que les autres constructions soient les mêmes." A somewhat more literal translation would be "and let the rest come to be from the same."

[21] The line referred to is the line from *D* parallel to the other asymptote, and its "end" is the end of the segment inside the hyperbola equal to the segment cut off between *D* and the hyperbola.

Let there be the things just mentioned, and set *EZ* equal to *DE*, and from *B* let line *BH* be drawn, if possible, parallel to *MN*. Therefore, *DE* is equal to *EH*. But this is absurd, for it was assumed that *DE* is equal to *EZ*.

Proposition 9

If from the same point two lines are drawn each cutting a section of a cone or circumference of a circle in two points, and [if] the segments cut off inside are divided [in the same ratio] as the wholes are to the segments cut off outside, so that the homologous lines are at the same point, [then] the line drawn through the points of division will intersect the section in two points, and [lines] drawn from the points of intersection to the point outside[22] will touch the curve.[23]

Let *AB* be a curve such as we have described[24] [fig. 8], and from a point *D* let *DE* and *DZ* be drawn cutting the curve at *Q*,*E* and at *Z*,*H* [respectively]. Furthermore, [let the ratio] that *EL* has to *LQ* be the same ratio that *DE* has to *QD*, and that that which *ZK* has to *KH* be the same that *DZ* has to *DH*. I say that [the line] joined from *L* to *K* will intersect the section at both [ends], and the lines joined from the points of intersection will touch the section.

For since *ED* and *ZD* both cut the section at two points, it is possible to draw a diameter of the section through *D*, and, with that, also [lines] touching [the section] on either side.[25] Let lines *DB* and *DA* be drawn touching [the section], and join *BA* not passing through *L*,*K*, if possible, but through [only] one of the two, or through neither.

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[22] The "point outside" is that from which the two cutting lines extend. Apollonius is not as careful here as he was in proposition 1 to say that this point is, indeed, outside the curve, although the rest of the enunciation makes that fact abundantly clear.

[23] Heath, among others, has said that IV.9 provides a method for drawing two tangents to a conic section from an exterior point (see *A History of Greek Mathematics*, II, 157–8). While this is true, it is not clear that providing such a method is Apollonius's intention. First, Apollonius does not present proposition IV.9 as a construction, and, second, in II.49, where Apollonius does give a method for drawing a tangent from a point outside a section, he emphasizes that, by the procedure given, two tangents may be drawn.

[24] Namely, a conic section or circumference of a circle.

[25] The second part of this statement, namely, that being possible to draw a diameter through *D* means that it is possible to draw tangents from *D*, could plausibly be related to *Conics* II.49 in which Apollonius shows how to draw a tangent from a point outside the section since, with one exception (*viz*., when the section is a

First, let it pass through L only and let it cut *ZH* at *M*. As *ZD* is to *DH*, therefore, so is *ZM* to *MH*. But this is absurd, for it has been assumed that as *ZD* is *DH* so is *ZK* to *KH*.

If *BA* passes through neither *L* [nor] *K*, then the absurdity occurs with regards to each line *DE,DZ*.

Proposition 10

The things [above] are common [to all the sections]. However, regarding the hyperbola only, if the other things are assumed, [and if] the [points of] intersection of the one line are contained by the [points of] intersection of the other, and [if] the point *D* is inside the angle contained by the asymptotes, the same things said above will happen, as we said above in proposition 2.[26]

Proposition 11

The same things being the case, if the points of intersection of one [of the lines] do not contain the intersection points of the other [line], then point *D* is in the angle contained by the asymptotes,[27] and the drawing (καταγραφὴ) and the proof will be the same is in [proposition] 9.

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hyperbola and *D* is in the angle adjacent to that contained by the asymptotes, so that one cannot draw a diameter from *D* to the section), Apollonius's construction of the tangent from a point *D* begins by drawing a diameter through *D*; the construction then follows from the fact that with *DA* as the diameter, *DT* the tangent, and *TK* drawn ordinatewise [see fig. 9], $DA = AK$ in the case of the parabola, and $BD{:}DA{::}BK{:}KA$ in the case of the hyperbola and ellipse. However, it is not enough just to be able to draw a diameter from point *D*, for if, as was pointed out regarding IV.1, the section is a hyperbola and *D* lies in the angle contained by the extension of the asymptotes beyond the center, then there is no tangent from *D* to the section while there is, indeed, a diameter through *D*. Nor does the mere fact that the two lines both cut the section in two points obviate all difficulties since drawing both a tangent and diameter from *D* demands that *D* be in the angle contained by the asymptotes, but, by *Conics* II.25, this will occur if two lines each meet a hyperbola at two points *and no point of meeting of either is contained by the points of meeting of the other*. For these reasons, it seems probable that, like the similar statement in IV.1, this statement is the work of a later and less careful writer (perhaps, the *same* author of the remark in IV.1) than Apollonius.

[26] In fact, since the point *D* lies in the angle contained by the asymptotes by assumption, one can apply the procedure in IV.2 twice, once for the first line and once for the second.

[27] This, as Heiberg notes, is clearly false.

Proposition 12

The same things being the case, if the points of intersection of one of the lines contain those of the other, and [if] the chosen point is in the angle adjacent to that contained by the asymptotes, [then] the line drawn through the [points of] division [and] extended will intersect the opposite section, and the lines drawn from the points of intersection to point D will touch the opposite sections.

Let *EH* be a hyperbola [see fig. 10], *NC,OP* the asymptotes, and *R* the center. Furthermore, let point *D* be in the angle *CRP*, let *DE,DZ* be drawn cutting the hyperbola each in two points, let *E,Q* be contained by *Z,H*, and let it be that as *ED* is to *DQ,EK* is to *KQ*, and that as *ZD* is to *DH,ZL* is to *LH*. It is to be shown that the [line] through *K* and *L* will intersect both [the section] *EZ* and also the opposite section, and the [lines] from the points of intersection to *D* will touch the sections.

Let *M* be the opposite section, and from *D* let *DM* and *DS* be drawn touching the section, let *MS* be joined, and, if possible, let it not pass through *K,L* but rather through [only] one of the two [points], or through neither.

First, let it pass through *K* and cut *ZH* at *X*. Therefore, as *ZD* is *DH,XZ* is to *XH* [III.37]. But this is absurd, for it has been assumed that as *ZD* is *DH,ZL* is to *LH*.

If *MS* passes through neither *K* nor *L*, then the impossibility occurs [with regards] to each [line] *ED* or *DZ*.

Proposition 13

The same things being the case, if the point *D* be on one of the asymptotes, and the remaining things are assumed to be the same, [then the line] drawn through the points of division will be parallel to the asymptote on which the point [is situated], and, extended, will intersect the section. Moreover, the [line] drawn from the point of intersection to the point [situated on the asymptote] will touch the section.

Let there be a hyperbola and [its] asymptotes, and let *D* be taken on one of the asymptotes [see fig. 11]. Let lines be drawn and divided as we have said above, and let a line *DB* be drawn from *D* touching the section. I say that [the line] drawn from *B* parallel to *PO* passes through *K,L*.

For if not, then surely it will pass through one [of these points] or through neither.

Let it pass through *K* only. Therefore, as *ZD* is to *DH*, *ZX* is to *XH* [III.35]. But this is absurd. Therefore the [line] drawn through *B* parallel to *PO* will not pass through *K* only. Therefore, it [will pass] through both [points].

Proposition 14

The same things being the case, if point *D* be on one of the asymptotes [see fig. 12], and *DE* [cuts the section] at two points, while *DH*, being parallel to the other asymptote, cuts the section at *H* only, and if it comes to be that as *DE* is to *DQ*, *EK* is to *KQ*, while *HL*, equal to *DH*, is placed in a line [with *DH*],[28] then the line drawn through points *K*,*L* will be parallel to the asymptote and will intersect the section, and [the line drawn] from [the point of] intersection to *D* will touch the section.

For, similar to what was said above,[29] drawing *DB* touching [the section], I say that [the line] drawn from *B* parallel to the asymptote *PO* will pass through points *K*,*L*.

Indeed, if it passes through *K* only, *DH* will not be equal to *HL*, which is absurd. And if [it passes] through *L* only, then it will not be that as *ED* is to *DQ*, *EK* is to *KQ*,[30] and if [it passes] neither through *K* nor through *L*, the absurdity will occur in both ways. Therefore, it will pass through both [points].

Proposition 15

If in opposite sections a point is taken between the two sections,[31] and [if a line] from the same touches one of the opposite sections, while another [line] cuts each of the opposite sections, and [if the ratio] that [the line] between the point and the one section, which the [first] line does not touch, has to the line between the point and the other section is the same as that that a line greater than [the

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[28] That is, *DH* is extended to *L* such that $DH = HL$.

[29] In proposition IV.13.

[30] As it must be according to *Conics* III.35.

[31] Note that Apollonius is referring to the opposite sections explicitly as *two* sections here.

one] between the sections has to its excess over the latter, set in a line [with it] and with the homologous lines being at the same extremities,[32] [then the line] drawn from the extremity of the greater line to the point of contact will intersect the section, and the [line] drawn from the point of intersection to the [original] point taken will touch the section.

Let there be opposite sections *A,B* [see fig. 13] and let some point *D* be taken between the sections [and] in the angle contained by the asymptotes, and from this [point] let *DZ* be drawn touching [the section] while *ADB* [be drawn] cutting the section. Furthermore, let *AG* have to *GB* the ratio that *AD* has to *DB*. It is to be shown that [the line] extended from *Z* to *G* will intersect the section, and [the line] drawn from the point of intersection to *D* will touch the section.

For since *D* is in the angle containing the section, it is possible to draw from *D* another [line] touching the section [II.49]. Let *DE* be drawn, let *ZE* be drawn and let it pass through, if possible, not through *G*, but through *H*. It will be then that as *AD* is to *DB,AH* [will be] to *HB* [III.37], which is absurd. For it was assumed that as *AD* is to *DB,AG* is to *GB*.

Proposition 16

The same things being the case, let *D* be a point in the angle adjacent to that contained by the asymptotes [see fig. 14], and let the remaining [constructions] be the same.

I say that the line joining *Z* to *G* will, [when] extended, intersect the opposite section, and [the line] from the point of intersection to *D* will touch the opposite section.

For let the same things be [as before], and let point *D* be in the angle adjacent to that contained by the asymptotes, and let *DE* be drawn from *D* touching section *A*, let *EZ* be joined and, [when] extended, let it not pass through *G*, but through *H*, if that is possible. It will be then that as *AH* is to *HB,AD* will be to *DB*, which is absurd. For it was assumed that as *AD* is to *DB,AG* is to *GB*.

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[32] Referring to the diagram [fig. 13], the lines *AD,DB* are said to have the same ratio as *AG,GB* such that the *corresponding* lines are those with the *same* endpoint, in other words, that the proportion given is *AD:DB::AG:GB* and *not*, say, *AD:DB::GB:AG*.

Proposition 17

The same things being the case, let point *D* be on an asymptote [see fig. 15].

I say that [the line] drawn from *Z* to *G* will be parallel to the asymptote on which the point [*D* is situated].

Let there be the same things as before, let point *D* be on one of the asymptotes, let [a line] be drawn through *Z* parallel [to the asymptote], and, if possible, let it not fall on *G*, but on *H*. It will be then that as *AD* is to *DB,AH* will be to *HB* [III.36],[33] which is absurd. Therefore, [the line] from *Z* parallel to the asymptote will fill on *G*.

Proposition 18

If in opposite sections a point is taken between the sections and from [that point] two lines are drawn cutting each of the sections, and if, as the [lines] between one of the sections [and the point][34] are to those between the other section and the same point, so are lines greater than those cut off between the opposite section[35] to their excess over the latter, [then] the line drawn through the extremities[36] of the greater lines will intersect the sections, and the lines drawn from the point of intersection to the [original] point taken will touch the curves.

Let there be opposite sections *A,B* [see fig. 16], and let point *D* be between the sections. Let it be assumed first that [*D*] be in the angle contained by the asymptotes, and through *D* let *ADB,GDQ* be

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[33] *Conics* III.36: "With the same things being so [namely, if some point be taken on one of the asymptotes of an hyperbola, and a straight line from it touch the section, and through the point of contact a parallel to the asymptote is drawn], if the straight line drawn across from the point neither cuts the section at two points nor is parallel to the asymptote it will meet the opposite section, and as the whole straight line is to the straight line between the section and the parallel through the point of contact, so will the straight line between the opposite section and the asymptote be to the straight line between the asymptote and the other section."

[34] The point being that taken between the sections, and the lines, those drawn through the point.

[35] These lines are the extension of the original two lines. Apollonius does not use the phrase ἐπ' εὐθείας here, as he does in IV.15, probably because he takes this to be obvious after the previous constructions.

[36] The extremities are those on the side where these lines exceed the lines between the sections, that is, the extremities located inside the section.

drawn. *AD*, therefore, is greater than *DB*, while *GD* [is greater] than *DQ*, since *BN* is equal to *AM* [II.16].[37] Furthermore, let *AK* have to *KB* the ratio that *AD* has to *DB*, and let *GH* have to *HQ* the ratio that *GD* has to *DQ*. I say that [the line] through *K,H* intersects the sections, and the [lines] from *D* to the points of intersection[38] will touch the section.

For since *D* is inside the angle contained by the asymptotes, it is possible to draw two [lines] touching [the section] [II.49]. Let *DE,DZ* be drawn,[39] and let *EZ* be joined. It will, thus, pass through points *K,H*.[40] For if [it passes through] one of these [points] only the other line will be cut in the same ratio by another point,[41] which is impossible. If [it passes] through neither [point], the same impossibility will occur in both lines.

Proposition 19

Let point *D* be taken then in the angle adjacent to that contained by the asymptotes, and let lines be drawn cutting the section and divided as said [above].

I say that [the line] produced through *K,H* will intersect each of the opposite sections [see fig. 17], and [the lines] from the points of intersection to [point] *D* will touch the section.

For let *DE,DZ* be drawn from *D* touching each of the sections. Therefore [the line] through *E,Z* will pass through *K,H*. For if not, it will surely go through one of the two, or through neither, and again one will similarly infer [from this] an absurdity.[42]

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[37] *Conics* II.16: "If in opposite sections some straight line is drawn cutting each of the straight lines containing the angle adjacent to the angles containing the sections, it will meet each of the opposite sections in one point only, and the straight lines cut off on it by the sections from the asymptotes will be equal."

[38] It is not entirely clear why Apollonius changes the order here, that is, from drawing lines from the points of intersection to *D*, as it is stated in the enunciation, to drawing lines from *D* to the points of intersection; mathematically, of course, the two formulations are equivalent.

[39] *DE* and *DZ* are understood to be the two lines touching the section just referred to.

[40] I am following Heiberg here in eliminating the next sentence in the Greek text, which reads "For if not, either it will pass through one of these [points] only, or through neither." It is possible that the interpolator was trying to strengthen the connection between this and propositions IV.9,12,13 where this same phrase appears.

[41] That is, two points will cut the same line segment into the same ratio.

[42] By *Conics* III.39, which states that "If two straight lines touching opposite

Proposition 20

If the point be taken on an asymptote, and the remaining things are the same, then the line drawn through the extremities of the [lines] exceeding[43] will be parallel to the asymptote on which the point is [situated], and the line drawn from the point to the point of intersection of the section and the line drawn through the extremities [of the lines exceeding] will touch the section.

Let there be opposite sections *A,B* [see fig. 18], and let point *D* be on one of the asymptotes, and let the remaining things be the same. I say that [the line] through *K,H* intersects the section, and [the line] from the point of intersection to *D* will touch the section.

Let *DZ* be drawn from *D* touching [the section], and a line be drawn from *Z* and parallel to the asymptote on which *D* is [situated]. It will then pass through *K,H.* For if not, it will either pass through one [of the two] or neither, and the same absurdities will occur as before.[44]

Proposition 21

Again let there be opposite sections *A,B* [see fig. 19], and point *D* on one of the asymptotes, let *DBK*, being parallel to one [of the two] asymptotes, meet the section at one point only, *B*,[45] but let *GDQ* meet both of the sections. Furthermore, let *GH* be to *HQ* as *GD* is to *DQ*, and let *DB* be equal to *BK.*

I say that [the line] through the points *K,H* will intersect the section and will be parallel to the asymptote on which point *D* is

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sections meet, and a straight line is produced through the points of contact, and a straight line drawn from the point of meeting of the tangents cuts both of the sections and the straight line joining the points of contact, then as the whole straight line drawn across is to the straight line cut off outside between the section and the straight line joining the points of contact, so will the segments of the straight line produced by the segments and the point of meeting of the tangents be to each other."

[43] Recall, according to the construction in IV.18, to which IV.20 still refers, lines have been drawn through *D* and extended into the section; the "lines exceeding" are the parts of the former lines that are inside the section, that is, which *exceed* the part between the opposite sections.

[44] By *Conics* III.36.

[45] Indeed, this must be the case by *Conics* II.13: "If in the place bounded by the asymptotes and the section some straight line is drawn parallel to one of the asymptotes, it will meet the section in one point only."

[situated], and that [the line] drawn from the point of intersection to *D* will touch the section.

For let *DZ* be drawn touching [the section], and let a line be drawn from *Z* parallel to the asymptote on which *D* is [situated]. It will thus pass through *K,H*. For if not, the absurdity spoken of before will occur.[46]

Proposition 22

Similarly, let there be opposite sections and [their] asymptotes and let *D* be similarly taken [see fig. 20].[47] Let *GDQ* [be taken] cutting the sections, while *DB* [be taken] parallel to one [of the two] asymptotes. Moreover, as *GD* is to *DQ*, [let] *GH* be to *HQ*, and [let] *BK* be equal to *DB*.

I say that [the line] through *K,H* will intersect each of the opposite sections, and the lines from the points of intersection to *D* will touch the section.

Let *DE,DZ* be drawn touching [the section], let *EZ* be joined, and, if possible, let it not pass through *K,H*, but through one [of the two points] or neither. If, on the one hand, [it passes] through *H* only, *DB* will not be equal to *BK* but to some other [line], which is absurd [III.31].[48] If, on the other hand, [it passes] through *K* only, it will not be that as *GD* is to *DQ,GH* is to *HQ*, but, some [line] to some other [line] [III.36]. If yet [it passes] through neither [of the points] *K,H*, then both impossibilities will occur.

Proposition 23

Again let there be opposite sections *A,B* [see fig. 21], and let point *D* be in the angle adjacent to that contained by the asymptotes. Let

[46] By *Conics* III.36.

[47] This seems to imply that *D* be taken on the asymptote; as Heiberg and others have pointed out, this part of the enunciation should read, ". . . and, similarly, let *D* be taken in the angle adjacent to that contained by the asymptotes." Halley, in his edition, makes the appropriate correction.

[48] *Conica* III.31: "If two straight lines touching opposite sections meet, and a straight line is produced through the points of contact, and through the point of meeting a straight line is drawn parallel to the asymptote and cutting both the section and the straight line joining the points of contact, then the straight line between the point of meeting and the straight line joining the points of contact will be bisected by the section."

BD be drawn cutting section *B* at one point only, [and thus] parallel to one [of the two] asymptotes, while *DA* is drawn similarly to section *A*,[49] and let *DB* be equal to *BH*, and *DA* to *AK*.

I say that the [line] through *K,H* meets the sections and [the lines] drawn from the points of intersection to *D* will touch the sections.

Let *DE,DZ* be drawn touching [the sections], let *EZ* be joined, and, if possible, let it not pass through *K,H*. So, either it will pass through one [of the two points] or through neither [of them], and either *DA* will not equal *AK* [III.31], but some other [line], which is absurd, or *DB* will not equal *BH*, or neither will equal neither, and again the same absurdity will occur in both [situations]. Therefore, *EZ* will pass through *K,H*.[50]

Proposition 24

A section of a cone will not meet a section of a cone or a circumference of a circle in such a way that part [of them] will be the same while [another] part will not be common.[51]

For, if possible, let the section of a cone *DABG* meet the circumference of the circle *EABG* [see fig. 22], let the same part of these, *ABG*, be common, and let *AD* and *AE* not be common. Let point *Q* be taken on this part,[52] let *QA* be joined, and through an arbitrary point *E*[53] draw [line] *DEG* parallel to *AQ*. Moreover, divide *AQ* in two [equal parts] at *H*, and through *H* draw diameter *BHZ*. Therefore, the line through *B* parallel to *AQ* touches each of the sections, and also will be parallel to *DEG*.[54] Also, in the one section *DZ* will be equal to *ZG*, while in the other *EZ* [will be equal] to *ZG* [I.46–47],[55] so that, also, *DZ* and *ZE* are equal, which is impossible.

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[49] That is, *DA* cuts *A* at one point only and is parallel to the second asymptote.

[50] With this, the first part of Book IV containing the various converses to *Conics* III.30–39 is finished. Now the book begins the treatment of its proper subject, the meeting of conic sections with one another.

[51] This is restated in *Conics* VI.6 as follows: "If there is a conic section, and part of it coincides with another part of another section so as to fit on it, then the [first] section is equal to the [second] section [*i.e.* they coincide throughout]."

[52] On *ABG*.

[53] E is an arbitrary point in the part *AE* or *AD*, that is, in the parts of the sections which do not coincide.

[54] The first conclusion, that the line through *B* parallel to *AQ* touches the section, is by Conica I.32, while the second, that the line through *B* will be parallel to *DEG* is by Euclid's *Elements* I.30.

[55] *Conics* I.46 states: "If a straight line touching a parabola meets the diameter,

Proposition 25

A section of a cone does not cut a section of a cone or circumference of a circle at more than four points.

For, if possible, let them cut at five [points] *A,B,G,D,E* [see fig. 23], and let the points of intersection *A,B,G,D,E* be taken in succession [so that] no point [of intersection] between them is left out, and let *AB,GD* be joined and produced. So, these will intersect outside the sections [in the cases of] the parabola and hyperbola [II.24–25].[56] Let them intersect at *L*, and let the ratio that *AL* has to *LB* be that which *AO* has to *OB*, while the ratio that *DL* has to *LG* be that which *DP* has to *PG*. Therefore, [the line] joined from *P* to *O* and produced will intersect the section on each side, and the lines joining the points of intersection and L will touch the section [IV.9]. So let it intersect at *Q,R* and let *QL,LR* be joined. Hence, they touch [the section]. Therefore, seeing that there is no point of intersection between *B,G*, [the line] *EL* cuts each of the sections.[57] Let it cut at *M,H*. Therefore, in one of the sections, *EN* will be to *NH* as *EL* is to *LH*, while in the other, *EN* will be to *NM* as *EL* is to *LM*. [But] this is absurd,[58] so that also [what was assumed] at the start is [absurd].

If *AB,DG* be parallel [see fig. 25], the sections will, of course, be ellipses or a circumference of a circle. Let *AB,GD* be cut in two [equal parts] at *O,P*, and let *OP* be joined and produced on each side. It will, then, intersect the sections. So, let it intersect at *Q,R*. *QR* will then be a diameter of the sections, and *AB,GD* set up

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then the straight line drawn through the point of contact parallel to the diameter in the direction of the section bisects the straight lines drawn in the section parallel to the tangent," while *Conics* I.47 states the analogous fact for an hyperbola, ellipse, and circumference of a circle, namely, "If a straight line touching an hyperbola or ellipse or circumference of circle meets the diameter, and if through the point of contact and the center a straight line is drawn in the direction of the section, then it bisects the straight lines drawn in the section parallel to the tangent."

[56] By *Conics* II.24 for the parabola and II.25 for the hyperbola. These state: "If two straight lines meet a parabola/hyperbola each at two points, and if a point of meeting of neither one of them is contained by the points of meeting of the other, then the straight lines will meet each other outside the section (but [in the case of the hyperbola] within the angle containing the section)."

[57] There is an implicit dependence here on the convexity (to use a slightly anachronistic term) of the conic sections proven, essentially, in *Conics* I.10. In curves that are not convex, one might well have a situation such as that in fig. 24, where EL does not pass through the segment BG.

[58] Since $EL{:}LH < EL{:}LM$, whereas $EN{:}NH > EN{:}NM$.

ordinatewise.[59] Let *ENMH* be drawn from *E* parallel to *AB,GD*. Therefore, *EMH* cuts *QR* and each of the curves, because there are no other intersections besides *A,B,G,D*.[60] In one of the sections, then, *NM* will equal *EN*, while in the other, *NE* will equal *NH*, so that *NM* equals *NH*, which is absurd.

Proposition 26

If, of the curves mentioned above,[61] some touch at one point, then they will not meet each other in more than two other points.

For let two of the [above] mentioned curves touch at point *A* [see fig. 26]. I say that they will not meet in more that two other points.

For if possible let them meet at *B,G,D*, and let points of intersection be taken in succession with no [point of intersection] between them being left out. Let *BG* be joined and produced, and, from *A*, let *AL* be drawn touching [the section]. [*AL*] will, thus, touch the two sections and intersect *GB*. Let it meet [*GB*] at *L*, and let it be that as *GL* is to *LB,GP* is to *PB*. Let *AP* be joined and produced. It will, thus, intersect the sections and [the lines drawn] from the points of intersection to L will touch the section [IV.1]. Let it be produced and intersect at *Q,R*, and let *QL,LR* be joined. These will touch the section. Therefore, [the line] joining *D* to *L* will cut each of the sections, and the earlier mentioned absurdity will occur.[62] Therefore, [the sections] will not cut one another at more than two points.

If, in an ellipse or circumference of a circle, *GB* be parallel to *AL*, the demonstration will be similar to that given above[63] once *AQ* is shown to be a diameter.

Proposition 27

If, of the curves mentioned above, some touch one another at two points, they will not meet one another at another [point].

For let two of the curves mentioned above touch one another at

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[59] "... τεταγμένως δὲ ἐπ'αὐτὴν κατηγμέναι αἱ *AB,ΓΔ*."

[60] The intention must be that there are no other points of intersection through which a line from *E* might pass.

[61] A section of a cone or circumference of a circle.

[62] That in the first part of IV.25.

[63] Namely, the demonstration in the second part of IV.25.

two points *A,B* [see fig. 27]. I say that they will not meet one another at another point.

For. If possible, let them meet also at *G*, and, to start, let *G* be outside the points of contact *A,B*,[64] and let lines be drawn from *A,B* touching [the sections]. Therefore, they will touch both of the curves. Let them touch and be produced to *L*, as in the first drawing,[65] and let *GL* be drawn. It cuts, then, each of the sections. Let it cut them at *H,M*, and let *ANB* be joined. Therefore, in one of the sections *GN* will be to *NH* as *GL* is to *LH*, while in the other, *GN* will be to *NM* as *GL* is to *LM* [III.37], which is absurd.

Proposition 28

If *GH* be parallel to [the lines] touching [the section] at points *A,B* [see fig. 28], as in the ellipses in the second drawing,[66] [then] joining *AB* we conclude that it is a diameter [II.27],[67] so that each of [the lines] *GH* and *GM* are bisected at *N*, which is absurd. Therefore, the curves do not meet one another at another point, but only at *A,B*.

Proposition 29

Let *G* be between the points of contact [see fig. 29], as in the third drawing.[68]

It is evident that the curves do not touch one another at *G* since it has been assumed that the curves touch at two [points] only. Indeed, let them cut one another at *G*. Let [lines] *AL,LB* be drawn from *A,B* touching the sections, let *AB* be joined and bisected at *Z*. Therefore, [the line drawn] from *L* to *Z* will be a diameter. [The diameter] will surely not pass through *G*. For if it did pass through, [then the line] drawn through *G* parallel to *AB* will touch each of

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[64] Apollonius means, of course, that *G* is not to be in the segment bounded by *A* and *B*.

[65] The first diagram in IV.25.

[66] The second diagram in IV.25.

[67] *Conics* II.27: "If two straight lines touch an ellipse or circumference of a circle, and if the straight line joining the points of contact is through the center of the section, the tangents will be parallel; but if not, they will meet on the same side of the center." Of course, the second part of the enunciation ("but if not . . .") is equivalent to the converse of the first part, and it is this that Apollonius is applying in IV.28.

[68] As is point B in relation to points R and A in IV.26.

the sections [II.5–6],[69] and this is impossible.[70] So, from *G* let *GKHM* be drawn parallel to *AB*. In the one section then *GK* will be equal to *KH*, while in the other *KM* [will be] equal to *KG*,[71] so that *KM* is equal to *KH*, which is impossible.

Similarly, if [the lines] touching [the sections] are parallel, the impossibility will be proven in the same way as above.[72]

Proposition 30

A parabola cannot touch a parabola at more points than one.

For, if possible, let parabolas *AHB*,*AMB* touch at *A*,*B* [see fig. 30], and let [lines] *AL*,*LB* be drawn touching [the parabolas]. They will, thus, touch both sections and will intersect at *L*. Let *AB* be joined and bisected at *Z*, and let *AZ* be drawn. Now, since the two curves *AHB*,*AMB* touch one another at *A*,*B*, they will not meet each other at another [point] [IV.27–29], so that *LZ* cuts each of the sections. Let it cut at *H*,*M*. In the one section, then, *LH* will be equal to *HZ* [I.35],[73] while, in the other, *LM* will equal *MZ*, which is impossible. Therefore, a parabola cannot touch a parabola at more points than one.

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[69] By *Conics* II.5 for the parabola or hyperbola and II.6 for the ellipse and circle. These state that "If the diameter of a parabola or hyperbola (ellipse or circumference of a circle) bisects some straight line (not through the center [in the case of the ellipse or circle]), the tangent to the section at the end of the diameter will be parallel to the bisected straight line."

[70] This is impossible because the curves were shown to cut and not touch one another at *G*. As in IV.25, this again depends on the convexity of the curves. For if the curves were not necessarily convex they might have a point where the tangent cuts the curve, or to use an anachronistic term, that they might have an inflection point at *G*, in which case the curves could have a common tangent at *G*, yet also intersect one another at *G*. I might add, that besides some among Nicomedes's conchoids and Perseus's spiric curves, I know of no other curves having inflection points that Apollonius might have known. Moreover, it is not at all clear how thoroughly these curves were investigated. In particular, it is not clear that they were shown to have, in some cases, a tangent that cuts the curve.

[71] By the definition of the diameter given in Book I (def.4).

[72] As in IV.28. This statement is unnecessary since point *G must*, in this case, be in the segment bounded by *A*,*B*; the proof would not be similar, but *identical* to IV.28.

[73] *Conics* I.35: "If a straight line touches a parabola, meeting the diameter outside the section, the straight line drawn from the point of contact ordinatewise to the diameter will cut off on the diameter beginning from the vertex of the section a straight line equal to the straight line between the vertex and the tangent, and no straight line will fall into the space between the tangent and the section."

Proposition 31

A parabola falling outside [a hyperbola] will not touch the hyperbola at two points.

Let there be a parabola *AHB* and a hyperbola *AMB* [see fig. 31], and, if possible, let [them] touch at *A,B*. Let [lines] be drawn from *A,B* touching each of the sections [that touch at] *A,B*, and let [the lines] intersect at *L*. Let *AB* be joined and bisected at *Z*, and let *LZ* be joined.

Now, since the sections *AHB,AMB* touch at *A,B*, they will not meet at another [point]; therefore, *LZ* cuts the sections at one and then another [point]. Let it cut [them] at *H,M*, and produce *LZ*. It will fall on the center of the hyperbola [II.29].[74] Let *D* be the center. By [the properties of] the hyperbola, as *ZD* is to *DM,MD* is to *DL*, and the remainders *ZM* to *ML* [I.37].[75] Therefore, *ZM* is greater than *ML*. But by [the properties of the parabola, *ZH* is equal to *HL*, which is impossible.[76]

Proposition 32

A parabola falling inside [an ellipse or circumference of a circle] will not touch the ellipse or circumference of the circle at two points.

For let there be an ellipse or circumference of a circle *AHB* and a parabola *AMB* [see fig. 32], and, if possible, let them touch at two [points] *A,B*, and let [lines] be drawn from *A,B* touching the sections and intersecting at *L*. Let *AB* be joined and bisected at *Z*, and

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[74] *Conics* II.29: "If in a section of a cone or circumference of a circle two tangents meet, the straight line drawn from their point of meeting to the midpoint of the straight line joining the points of contact is a diameter of the section." Therefore, *LZ* is a diameter, and, therefore, it falls on the center of the hyperbola.

[75] *Conics* I.37: "If a straight line touching an hyperbola or ellipse or circumference of a circle meets the diameter, and from the point of contact to the diameter a straight line is dropped ordinatewise, then the straight line cut off by the ordinate from the center of the section with the straight line cut off by the tangent from the center of the section will contain an area equal to the square on the radius of the section [*i.e.* the line from the center to the hyperbola or ellipse], and with the straight line between the ordinate and the tangent will contain an area having the ratio to the square on the ordinate which the transverse has to the upright." Referring to fig. 31, the first part of this proposition states that rect.*ZD,DL* = *sq.DM*, from which it follows that *ZD:DM::DM:DL*, also, *ZD:DM::DM:DL::(ZD – DM):(DM – DL)::ZM:ML*.

[76] Here we are using the fact that the parabola lies outside the hyperbola, for this means that *HZ* > *MZ*. So, *MZ* > *ML* > *HL* implies that *HZ* > *HL*, but *HZ* = *HL*.

let *LZ* be joined. [*LZ*] will cut each section at one [point] then another, as we said [above]. Let it cut [them] at *H,M*, and let *LZ* be produced to *D*, and let *D* be the center of the ellipse or circle. Therefore, by [the properties of] the ellipse and circle, as *LD* is to *DH,DH* is to *DZ*, and the remainders *LH* to *HZ* [I.37]. And *LD* is greater than *DH*. Therefore, *LH* is greater than *HZ*. But, by [the properties of] the parabola *LM* is equal to *MZ*, which is impossible.

Proposition 33

A hyperbola will not touch a hyperbola, having the same center, at two points.

For let hyperbolas *AHB,AMB*, having the same center *D* [see fig. 33], touch at *A,B*, if possible. Let [lines] *AL,LB* be drawn from *A,B* touching [the hyperbolas] and intersecting one another, and let *DL* be joined and produced.

Moreover, let *AB* be joined. Therefore, *DZ* bisects *AB* at *Z*.[77] *DZ* then cuts the sections at *H,M*. By [the properties of] the hyperbola *AHB*, [the rectangle contained] by *ZD,DL* will be equal to [the square] on *DH*, while by [the properties of] the hyperbola *AMB*, [the rectangle contained] by *ZD,DL* will be equal to the [square] on *DM* [I.37]. Therefore, [the square] on *MD* is equal to [the square] on *DH*, which is impossible.

Proposition 34

If an ellipse touches an ellipse or circumference of a circle, having the same center, at two points, [then the line] joining the points of contact falls on the center.

Let the above mentioned curves touch one another at points *A,B* [see fig. 34]. Let *AB* be joined, and let [lines] touching the sections be drawn through *A,B*, and, if possible, intersecting at *L*. Let *AB* be bisected at *Z*, and let *LZ* be joined. Therefore, *LZ* is a diameter of the sections [II.29].

If possible, let the center be *D*. Therefore, [the rectangle contained] by *AD,DZ* will be equal to [the square] on *DH* by [the properties

[77] *DZ* is *DL* produced to *Z*; it bisects *AB* since, by *Conics* II.30, *DL* is a diameter and *AB* is drawn ordinatewise.

of] the one section, but to [the square] on *MD* by [the properties of] the other, so that [the square] on *HD* is equal to [the square] on *DM* [I.37], which is impossible. Therefore, [the lines] from *A,B* touching [the sections] do not intersect. Therefore, they are parallel, and for the same [reason], *AB* is a diameter [II.27], so that it falls on the center, which was to be proved.

Proposition 35

A section of a cone or circumference of a circle will not intersect a section of a cone or circumference of a circle, not having its convexity in the same [direction],[78] at more than two points.

For if possible let a section of a cone or circumference of a circle *ABG* meet a section of a cone or circumference of a circle *ADBEG* [see fig. 35], not having its convexity in the same [direction], at more points than two, *A,B,G.*

Since three points *A,B,G* have been taken on the curve *ABG*, [if] *AB,BG* are joined, they will contain an angle having its concavity in the same [direction][79] as that of the curve *ABG.* For the same [reason] *ABG* contain an angle[80] whose concavity is in the same [direction] as that of curve *ADBEG.* Therefore, the curves we have been speaking of have both their concave and convex parts in the same direction, which is impossible.

Proposition 36

If a section of a cone or circumference of a circle intersects one of the opposite sections at two points, and the curves between the points of intersection have their concavity in the same [direction], [then] the curve, being produced at the points of intersection, will not intersect the other opposite section.

Let there be [see fig. 36] opposite sections *D,AEGZ*, and let there be a section of a cone or circumference of a circle *ABZ* intersecting

[78] Literally, "not having its convexities towards the same parts (μὴ ἐπὶ τὰ αὐτὰ μέρη τὰ κυρτὰ ἔχουσα)."

[79] ". . . ἐπὶ τὰ αὐτὰ τοῖς κοίλοις . . ."

[80] That is, "the lines *AB,BG* contain an angle . . ."; Halley has "*AB,BΓ*" in his text in place of "*ABΓ*."

one [of the two] opposite sections at two points *A,Z*, and let the sections *ABZ,AGZ* have their concavity in the same [direction]. I say that *ABZ* produced will not intersect [section] *D*.

For let *AZ* be joined. Since *D,AGZ* are opposite sections and line *AZ* cuts a hyperbola in two [points], produced, it will not intersect the opposite section *D* [II.33].[81] Neither, therefore, will the curve *ABZ* intersect [section] *D*.[82]

Proposition 37

If a section of a cone or circumference of a circle intersects one of the opposite sections, it will not intersect the remaining [section] at more points than two.

Let there be opposite sections *A,B* [see fig. 37], and let a section of a cone or circumference of a circle *ABG* meet section *A*, and let [*ABG*] cut the opposite section *B* at [points] *B,G*. I say that it will not intersect *BG* at another point.

For if possible, let it intersect [*BG*] at *D*. Therefore, *BGD* meets section *BG*, having its concavity in the same direction, at more points than two, which is impossible [IV.35].

This will be shown similarly if curve *ABG* touches the opposite section.

Proposition 38

A section of a cone or circumference of a circle will not intersect opposite sections at more points than four.

This is evident from [the fact] that intersecting one of the opposite sections it cannot intersect the remaining section at more than two [points] [IV.37].

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[81] *Conics* II.33: "If a straight line meeting one of the opposite sections, when produced both ways, falls outside the section, it will not meet the other section, but will fall through the three places of which one is that contained by the angle containing the section, and two are those contained by the angle adjacent to the angle containing the section."

[82] Since, by *Conics* I.10, section *ABZ* will not meet *AZ* at any other point besides *A* and *B*.

Proposition 39

If a section of a cone or circumference of a circle touches one of the opposite sections in the concave part of the latter,[83] it will not intersect the other opposite section.

Let there be opposite sections *A,B* [see fig. 38], and let *GAD* touch section *A*. I say that *GAD* will not intersect *B*.

Let *EAZ* be drawn from [point] *A* touching [the section]. [Then] it touches each of the sections at [point] *A*, hence, it will not intersect *B*,[84] so that neither will *GAD* intersect *B*.

Proposition 40

If a section of a cone or circumference of a circle touches each [of the two] opposite sections at one point, it will not intersect the opposite sections at another [point].

Let there be opposite sections *A,B* [see fig. 39], and let a section of a cone or circumference of a circle touch each [of the sections] *A,B* at [points] *A,B*. I say that the curve *ABG* will not intersect the sections *A,B* at another point.

Indeed, since curve *ABG* touches section A and meets (συμπίπτουσα) *B* at one [point], it will not, therefore, touch *A* in [the direction of] its concavity. Similarly, it will be shown that neither [will it touch] B in [the direction of] its concavity. Let *AD,BE* be drawn touching sections *A,B*; these, then, will touch the curve *ABG*. For, if possible, let one of them cut [the curve], and let it be *AZ*. Therefore, between *AZ*, touching [the section *A*], and the section *A*, a line *AH* has been interposed, which is impossible.[85] Therefore, it touches *ABG*, and, because of this, it is evident that *ABG* does not meet opposite sections at another point.

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[83] In other words, so that the tangent is on the same side of both sections.

[84] This follows from *Conics* II.3, which states that "If a straight line touches an hyperbola, it will meet both of the asymptotes . . ." For if *EAZ* is produced, it will, by II.3, fall in the two angles adjacent to that contained by the asymptotes, and, therefore, it cannot meet the opposite section.

[85] By *Conics* I.36, which, besides giving the main property of tangents to central conic sections, tells us that, ". . . another straight line will not fall into the space between the tangent and the section of the cone."

Proposition 41

If a hyperbola intersects one of the opposite sections in two points having its convexity turned oppositely, [then] the opposite section [of the hyperbola] will not intersect the other opposite section.

Let there be opposite sections *ABD,Z* [see fig. 40], let hyperbola *ABG* meet *ABD* at points *A,B*, the former having its convexity turned oppositely to the concavity of the latter,[86] and let *E* be the opposite section of *ABG*. I say that [*E*] will not intersect *Z*.

Let *AB* be joined and extended to *H*. Since, indeed, the line *ABH* cuts the hyperbola *ABD*, [and] extended it falls outside each section, it will not intersect section *Z* [II.33]. Similarly, because [line *ABH* cuts] the hyperbola *ABG*, it will not intersect opposite section *E*; neither, therefore, will *E* intersect *Z*.

Proposition 42

If a hyperbola intersects each [of two] opposite sections, its opposite section will intersect neither of the opposite sections at two points.

For let there be opposite sections *A,B* [see fig. 41], and let hyperbola *AGB* intersect each of the opposite sections *A,B*. I say that the opposite section of *AGB* will not intersect sections *A,B* at two points.

For, if possible, let it meet [one of the opposite sections] at *D,E*, and let *DE* be joined and produced. Because of the section *DE*,[87] line *DE* will not intersect section *AB* [II.33], while, because of section *AED*, it will not intersect [section] *B*, for it passes through three places,[88] which is impossible. Similarly it will be shown that [*AGB*] will not intersect *B* at two points.

For the same [reasons], neither will it touch either of the [opposite sections]. For drawing *QE* touching, it will touch each of the sections, so that, because of [section] *DE*, it will not intersect [section *AG*], whereas, because of [section] *AE* it will not meet [section] *B*, so that neither will *AG* meet *B*, which is contrary to what was assumed.

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[86] Literally, "having convex [parts] turned oppositely to concave [parts]."

[87] That is, because line *DE* passes through two points of section *DE*.

[88] These are the three places referred to in *Conics* II.33, namely, that contained by the angle containing the section, and those contained by the angle adjacent to the angle containing the section.

Proposition 43

If a hyperbola cuts each [of two] opposite sections at two points[89] having its convexity turned oppositely to each of them, the opposite section [of the hyperbola] will intersect neither of the opposite sections.

Let there be opposite sections *A,B* [see fig. 42], and let hyperbola *GABD* cut each [of the sections] *A,B* at two points containing convexities turned oppositely. I say that its opposite section *EZ* intersects neither [of the sections] *A,B*.

For, if possible, let it intersect [section] *A* at [point] *E*, and let *GA,DB* be joined and extended; these, then, will intersect one another [II.25]. Let them intersect at *Q*; *Q* will then be in the angle contained by the asymptotes of section *GABD*. And *EZ* is the opposite section of [*AB*]. Therefore, [the line] joining *E* to *Q* will fall in the angle contained by *AQB*.[90] Again since *GAE* is a hyperbola, and *GAQ,QE* intersect, and the intersection points *G,A* do not contain *E*, point *Q* will be between the asymptotes of section *GAE*.[91] And *BD* is the opposite section [of *GAE*]. Therefore, [the line] from *B* to *Q* falls inside the angle [contained] by *GQE*; which is impossible, for it also fell in the angle contained by *AQB*. Therefore, *EZ* will not intersect one [of the opposite sections] *A,B*.

Proposition 44

If a hyperbola cuts one [of two] opposite sections at four points, the opposite section [of the hyperbola] will not intersect the other [of the two] opposite sections.

Let there be opposite sections *ABGD,E* [see fig. 43], and let a hyperbola cut *ABGD* at four points, *A,B,G,D*, and let its opposite section be *K*. I say that *K* will not intersect *E*.

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89 Here, one can see clearly that Apollonius does not treat the opposite sections as just another conic section, for otherwise IV.25 would make this proposition gratuitous.

90 That is, *AQ,QB*.

91 As both Commandino and Halley observed, this conclusion does not follow from what has been stated. The problem is that one does not know what the fate of *QE* is after it intersects section *GAE*: if it intersects *GAE* at a second point or if it is tangent to *GAE* at *E* then *Q* will, indeed, be in the angle contained by the asymptotes of *GAE*, but if it intersects *GAE* only at *E*, and is not tangent, then *Q* need not be in the angle contained by the asymptotes.

For if possible, let it intersect at *K*. Let *AB,GD* be joined and produced; they will, then, intersect one another. Let them intersect at *L*, and let *AP* have to *PB* the ratio that *AL* has to *LB*, and let *DR* have to *RG* [the ratio] that *DL* has to *LG*. Therefore, the line produced through *P,R* will intersect the sections on each side, and [the lines] from *L* to the points of intersection will touch [the sections] [IV.9]. Let *KL* be joined and produced. It will cut the angle contained by *BLG* and the sections at one and then another point. Let it cut [them] at *Z,M*. By [the properties of] the opposite sections *AQZH,K*, as *NK* is to *KL,NZ* is to *ZL*, while by [the properties of the opposite sections] *ABGD,E*, as *NK* is to *KL,NM* is to *ML* [III.39], which is impossible. Therefore *E,K* will not intersect one another.

Proposition 45

If a hyperbola intersects [one of two opposite sections] at two points having its concavity in the same [direction] as that section, while it intersects [the other of the two opposite sections] at one point, then the opposite section [of the hyperbola] will intersect neither of the opposite sections.

Let there be opposite sections *AB,G* [see fig. 44], and let the hyperbola *AGB* intersect *AB* at [points] *A,B*, and [let it intersect the section] *G* at one [point] *G*, and let *D* be the opposite section of *AGB*. I say that *D* will intersect neither [of the sections] *AB,G*.

For let *AG,BG* be joined and produced. Therefore, *AG,BG* will not intersect section *D* [II.33]. Neither will they intersect section *G* at another [point] besides *G*. For if they meet [section *G*] at another point, they will not intersect the opposite section *AB* [II.33], whereas it is assumed that they do intersect. Therefore, lines *AG,BG* meet section *G* at one point *G*, while they do not meet *D* at all. Therefore, *D* will be [contained] in the angle *EGZ*, so that section *D* will not intersect *AB,G*.

Proposition 46

If a hyperbola meets one [of two] opposite sections at three points, the opposite section [of the hyperbola] will not intersect the other opposite section at more than one point.

Let there be opposite sections *ABG,DEZ* [see figs. 45, 46], and let hyperbola *AMBG* meet *ABG* at three points *A,B,G*, and let *DEK* be

the opposite section of *AMG*.[92] I say that *DEK* will not meet *DEZ* at more points than one.

For if possible let them meet at *D,E* and let *AB,DE* be joined.

Now, they will either be parallel or not.

To start, let them be parallel [see fig. 45], and let *AB,DE* be bisected at *H,Q*, and let *HQ* be joined; therefore, [*HQ*] is a diameter for all the sections, and *AB,DE* are set up ordinatewise. Let *GNCO* be drawn from *G* parallel to *AB*. It will, then, be drawn ordinatewise to the diameter and it will cut the sections, one and then the other. For if [it were to cut them] at the same [point], [the sections] would no longer meet at three [points], but four. In [section] *AMB*, then, *GN* [will be equal] to *NC*, while in *ALB*, *GN* will be equal to *NO*. And, therefore, *ON* is equal to *NC*, which is impossible.

So, let [lines] *AB,DE* not be parallel [see fig. 46], but, being produced, let them intersect at [point] *P*. Let *GO* be drawn parallel to *AP* and let it intersect *DP* produced at [point] *R*, and let *AB,DE* be bisected at *H,Q*, through *H,Q* let diameters *HSI,QLM* be drawn, while from *I,L,M* let *IUT,MU,LT* [be drawn] touching the sections; *IT*, then, will be parallel to *DP*, while *LT,MU* [will be] parallel to *AP,OR* [II.5]. Since, as [the square] on *MU* is to [the square] on *UI*, [the rectangle contained] by *AP,PB* is to [the rectangle contained] by *DP,PE* [III.19],[93] but, as [the rectangle contained] by *AP,PB* is to [the rectangle contained] by *DP,PE*, [the square] on *LT* is to [the square] on *TI*, and, therefore, as [the square] on *MU* is to [the square] on *UI*, [the square] on *LT* is to [the square] on *TI*. For the same reasons, as [the square] on *MU* is to [the square] on *UI*, [the rectangle contained] by *CR,RG* is to [the rectangle contained] by *DR,RE*, and as [the square] on *LT* is to [the square] on *TI*, [the rectangle contained] by *OR,RG* is to [the rectangle contained] by

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[92] The text adds here, "while *DEZ* [is the opposite section] of *ABG*." This is an obvious and unnecessary interpolation (and Heiberg marks it as such in his edition), since it has already been said that *ABG* and *DEZ* are opposite sections.

[93] *Conics* III.19: "If two straight lines touching opposite sections meet, and parallels to the tangents are drawn cutting each other and the section, then, as the squares on the tangents are to each other, so will the rectangle contained by the straight lines between the section and the point of meeting of the straight lines be to the rectangle contained by the straight lines taken similarly." Where I have written "[the rectangle contained] by "*AP,PB*," and "*DP,PB* the text has "*APB*" and "*DPB*." Similarly, where I have written "[the rectangle contained] by *CR,RG*" and "*DR,RE*," the text has "*CRG*" and "*DRE*." Unless noted otherwise, I shall continue this convention.

DR,RE. Therefore, [the rectangle contained] by *OR,RG* is equal to [the rectangle contained] by *CR,RG*, which is impossible.[94]

Proposition 47

If a hyperbola touches [one of two] opposite sections, while it cuts [the other] at two points, [then] the opposite section [of the hyperbola] will intersect neither of the opposite sections.

For let there be opposite sections *ABG,D* [see fig. 47] and let some hyperbola *ABD* cut *ABG* at *A,B*, and touch [section] *D* at [point] *D*, and let *GE* be the opposite section of *ABD*. I say that *GE* intersects neither [of the opposite sections] *ABG,D*.

For, if possible, let it intersect *AB* at *G*, and let *AB* be joined, and let [a line] be drawn through *D* touching [section *ABD*] and intersecting [line] *AB* at *Z*. Therefore, point *Z* will be inside [the angle contained by] the asymptotes of section *ABD* [II.25]. And *GE* is the opposite section of [*ABD*]. Therefore, the line from *G* to *Z* falls inside the angle contained by *BZ,ZD*. Again, since *ABG* is a hyperbola, and *AB,GZ* intersect, and the points of intersection *A,B*, do not contain *G*, point *Z* is between the asymptotes of section *ABG*.[95] And *D* is the opposite section [of *ABG*]. Therefore, [the line] from *D* to *Z* falls in the angle [contained by] *AZ,ZG*, which is absurd. For, it fell in [the angled contained by] *BZ,ZD*. Therefore, *GE* does not intersect one [of the opposite sections] *ABG,D*.

Proposition 48

If a hyperbola touches one [of two] opposite sections at one point, while it intersects it at two, [then] the opposite section [of the hyperbola] will not intersect the [other] opposite section.

Let there be opposite sections *ABG,D*, [see fig. 48] and let some hyperbola *AHG* touch [*ABG*] at [point] *A*, and let it intersect [*ABG*] at [points] *B,G*, and let *E* be the opposite section of *AHG*. I say that *E* will not intersect *D*.

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[94] For since sq.*MU*:sq.*UI*::sq.*LT*:sq.*TI*, we have rect.*CR,RG*:rect.*DR,RE*::rect.*OR,RG*: rect.*DR,RE*, and, therefore, by *Elem.* V.9, rect.*CR,RG* = rect.*OR,RG*, so that, by *Elem.*VI.1, *CR* = *OR*. This is impossible because *O* and *C* are different points, as Apollonius emphasizes at the start of the proof.

[95] As in IV.43, and for the same reasons, this is incorrect.

For, if possible, let it intersect at [point] *D*, let *BG* be joined and extended to *Z*, and let [line] *AZ* be drawn from *A* [touching the section]. As in the earlier [demonstration], it will be shown that point *Z* is inside that angle contained by the asymptotes. Moreover, *AZ* will touch both sections, and *DZ* produced will cut the sections at [points] *H,K* between [points] *A,B*. Let *GL* have to *LB* that ratio which *GZ* has to *ZB*, and let *AL* be joined and produced; it will cut the sections, one and then the other. Let it cut [them] at *N,M*. [The lines], therefore, from *Z* to *N,M* will touch the sections [IV.1], and, as in the earlier [demonstration], by [the properties of] the one section, *CK* is to *KZ* as *CD* is to *DZ*, while, by [the properties of] the other [section], *CH* is to *HZ* as *CD* is to *DZ* [III.39], which is impossible. Therefore, it does not intersect the opposite [section].

Proposition 49

If a hyperbola, touching one [of two] opposite sections, intersects the same at another [point], [then] the opposite section [of the hyperbola] will not intersect the other opposite section at more points than one.

Let there be opposite sections *ABG,EZH* [see figs. 49,50], and let some hyperbola *DAG* touch [*ABG*] at [point] *A*, and let it cut [*ABG*] at *G*, and let *EZQ* be the opposite section of *DAG*. I say that it will not intersect the other opposite section at more points than one.

For if possible let it meet at two [points] *E,Z*, and let *EZ* be joined, and through *A* let *AK* be drawn touching the sections.

Now, [*EZ* and *AK*] will be parallel or not [parallel].

To start, let them be parallel, and let the diameter bisecting *EZ* be drawn [see fig. 49]; therefore, it will pass through *A*, and it will be the diameter of the two conjugates [II.34].[96] Let [line] *GLDB* be drawn through *G* parallel to *AK,EZ*. Therefore, it will cut the sections at one point and then another. In the one [section] then *GL* will be equal to *LD*, whereas in the remaining [section] *GL* [will be equal] to *LB*. This is impossible.

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[96] *Conics* II.34: "If some straight line touch one of the opposite sections and a parallel to it be drawn in the other section, then the straight line drawn from the point of contact to the midpoint of the parallel will be a diameter of the opposite sections." Referring to the common diameter of the two opposite sections as the "diameter of *the two conjugates* (διάμετρος τῶν δύο συζυγῶν)," is unusual and somewhat misleading. Nevertheless, the phrase occurs again in IV.50,53.

So, let *AK,EZ* not be parallel [see fig. 50], let them intersect at *K*, and let *GD*, drawn parallel to *AK*, intersect *EZ* at *N*. Let *AM*, bisecting *EZ*, cut the sections at *C,O*, and let *CP,OR* be drawn from *C,O*, touching the sections. Therefore, as [the square] on *AP* is to [the square] on *PC*, so is [the square] on *AR* to [the square] on *RO*,[97] and, for this reason, as [the rectangle contained] by *DN,NG* is to [the rectangle contained] by *EN,NZ*, [the rectangle contained] by *BN,NG* is to [the rectangle contained] by *EN,NZ*.[98] Therefore, [the rectangle contained] by *DN,NG* is equal to [the rectangle contained] by *BN,NG*, which is impossible.

Proposition 50

If a hyperbola touches one [of two] opposite sections at one point, the opposite section [of the hyperbola] will not intersect the other opposite section at more points than two.

Let there be opposite sections *AB,EDH* [see fig. 51, 52] and let a hyperbola *AG* touch *AB* at *A*, the let *EDZ* be the opposite section of *AG*. I say that *EDZ* will not intersect *EDH* at more points than two.

For, if possible, let [*EDZ*] meet [*EDH*] at three [points] *D,E,Q*, let *AK* be drawn touching [sections] *AB,AG*, let *DE* be joined and extended, and, to start, let *AK,DE* be parallel [see fig. 51]. Let *DE* be bisected at *L*, and let *AL* be joined. *AL* will then be a diameter for the two conjugates and will cut the sections between *D,E*, at points *M,N*.[99] Let *QZH* be drawn from *Q* parallel to *DE*. In the one section, then, *QC* will equal *CZ*, while in the other, *QC* [will equal] *CH*, so that also *CZ* equals *CH*, which is impossible.

So let *AK,DE* not be parallel [see fig. 52], but let them intersect at [point] *K*, and let the remaining [constructions] be the same. Let *AK* be produced and let it meet *ZQ* at [point] *R*. As before, we will show that, in section *ZDE*, [the rectangle contained] by *ZR,RQ* is to [the square] on *RA* as [the rectangle contained] by *DK,KE* is to [the

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[97] Since *PC* and *OR* are parallel.

[98] By *Conics* III.19, sq.*AP*:sq.*PC*::rect.*DN,NG*:rect.*EN,NZ* and sq.*AR*:sq.*RO*:: rect.*BN,NG*:rect.*EN,NZ*, so that, since sq.*AP*:sq.*PC*::sq.*AR*:sq.*RO*, we have, rect.*DN,NG*: rect.*EN,NZ*::rect.*BN,NG*:rect.*EN,NZ*.

[99] Here, there is an interpolation which adds, "so that *DLE* is bisected at [point] *L*," but since this has *already* been said, the interpolation in fact adds nothing.

square] on *AK*, whereas in [section] *HDE*, [it is the rectangle contained] by *HR,RQ* [which] is to [the square] on *RA* [as the rectangle contained by *DK,KE* is to the square on *AK*] [III.19]. Therefore, [the rectangle contained] by *HR,RQ* is equal to [the rectangle contained] by *ZR,RQ*, which is impossible.[100] Therefore, *EDZ* does not meet *EDH* at more point than two.

Proposition 51

If a hyperbola touches each [of two] opposite sections, the opposite section [of the hyperbola] will intersect neither of the opposite sections.

Let there be opposite sections *A,B* [see fig. 53], and let hyperbola *AB* touch each of these at [points] *A,B*, and let the opposite section [of *AB*] be *E*. I say that *E* will intersect neither of [the sections] *A,B*.

For, if possible, let it intersect *A* at *D*, and let [lines] be drawn from [points] *A,B*, touching the sections; they will intersect one another in the angle [contained by] the asymptotes of section *AB*. Let them intersect at *G*, and let *GD* be joined. The [line] *GD*, therefore, will be in the place between *AG,GB*. But it is between *BG,GZ*,[101] which is absurd. Therefore, *E* does not intersect *A,B*.

Proposition 52

If each [of two] opposite sections touch each [of two] opposite sections at one point, each having its concavity in the same [direction], [then] they will not intersect at another point.

For let opposite sections touch one another at points *A,D* [see fig. 54]. I say that they will not meet at another point.

For, if possible, let them meet at *E*. Since, indeed, a hyperbola touching one of the opposite sections at *D* has intersected it at *E*, therefore, [section] *AB* will not meet [section] *AG* at more points than one [IV.49].[102] Let *AQ,QD* be drawn from *A,D* touching the

[100] Since, rect.*ZR,RQ*:sq.*RA*::rect.*HR,RQ*:sq.*RA*, rect.*ZR,RQ* = rect.*HR,RQ*, and, hence, *ZR* = *HR*, which is impossible.

[101] Since *BG* touches section *BZ*, it does not touch section *AD*, and, therefore, *DG* must fall within the angle contained by *GB* and the extension beyond *G* of *AG*.

[102] *AB* and *AG* are, of course, the opposite sections, respectively, of the hyperbola and of the branch of the opposite sections that the hyperbola meets.

sections, let *AD* be joined, let *EBG* be drawn through *E* parallel to *AD*, and let the second diameter *QKL* [*sic.*] of the opposite sections be drawn from *Q*. It will then bisect *AD* at *K*. And, therefore, *EB,EG* will be bisected at *L* [II.39].[103] Therefore, *BL* is equal to *LG*, which is impossible. Therefore, [the sections] will not intersect at another point.

Proposition 53

If a hyperbola touches one [of two] opposite sections at two points, the opposite section [of the hyperbola] will not intersect the other opposite section.

Let there be opposite sections *ADB,E* [see fig. 55], and let hyperbola *AG* touch *ADB* at two points *A,B*, and let *Z* be the opposite section of *AG*. I say that *Z* will not intersect *E*.

For, if possible, let it intersect at [point] *E*, and let *AH,HB* be drawn from *A,B*, touching the sections, let *AB* and *EH* be joined, and let [*EH*] be produced. It will cut the sections at one point and then another; let it be as *EHGDQ*. Since *AH,HB* indeed touch [the sections], and *AB* joins the points of contact, in one of the conjugate sections *QD* will be to *DH* as *QE* is to *EH*, while in the other, *QG* [will be] to *GH* as *QE* is to *EH* [III.39], which is impossible. Therefore, [section] *Z* does not meet [section] *E*.

Proposition 54

If a hyperbola touches one [of two] opposite sections, having its convexity turned oppositely, then the opposite section [of the hyperbola] will not intersect the other opposite section.

For let there be opposite sections *A,B* [see fig. 56], and let some hyperbola *AD* touch section *A* at [point] *A*, and let the opposite section of *AD* be *Z*. I say that *Z* will not intersect *B*.

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[103] *Conics* II.39: "If two straight lines meeting touch opposite sections, the straight line drawn through the center and the point of meeting of the tangents bisects the straight line joining the points of contact." Since *QLK* has been drawn as a second diameter, it must pass through the center of the sections, and, therefore, by II.39, bisect *AD*. This means, also, that *AD* is drawn ordinatewise with respect to diameter *QLK* (since, by *Conics* II.41, no other line through *K* can be bisected by *QLK*), so that *EBG*, being parallel to *AD*, is also drawn ordinatewise and, therefore, bisected by *QLK*.

Let [line] *AG* be drawn from [point] *A* touching the sections; therefore, because of [the properties of section] *AD,AG* will not intersect *Z*, while, because of [the properties of section] *A*, it will not intersect *B* [II.33], so that *AG* falls between sections *B,Z*. It is evident [then] that *B* will not intersect *Z*.

Proposition 55

Opposite sections will not intersect opposite sections at more points than four.

For let there be [one pair of] opposite sections *AB,GD*, and another [pair of] opposite sections *ABGD,EZ*, and, to start, let *ABGD* cut each of *AB,GD* at four points *A,B,G,D* containing convexities turned oppositely, as in the first drawing [see fig. 57]. Therefore, the opposite section of *ABGD*, that is, *EZ*, will not intersect *AB,GD* [IV.43].

But let *ABGD* cut *AB* at *A,B* and *G* at one [point] *G*, as in the second drawing [see fig. 58]. Therefore *EZ* does not intersect [section] *G* [IV.41]. If *EZ* meets *AB*, it will meet at one point only; for if it meets it at two points, its opposite section, *ABG*, will not meet the other opposite section *G* [IV.43]. But it has been assumed that [it] meets it at one point *G*.

If, as in the third drawing [see fig. 59], *ABG* cuts *ABE* at two [points] *A,B*, while *EZ* meets *ABE* [at one point], [*EZ*] will not intersect [section] *D* [IV.41], whereas, intersecting *ABE*, it will not intersect [*ABE*] at more points than two.[104]

If, as in the fourth drawing [see fig. 60], *ABGD* cuts each [of two opposite sections] at one point, *EZ* will intersect neither at two points. {So that by the things already said, and their converse, *ABGD,GZ* [*sic.*] will not meet the opposite sections *BE,EZ* [*sic.*] at more points than four.}[105]

If the sections have their concavities in the same [direction], and one cuts the other at four [points] *A,B,G,D*, as in the fifth drawing [see fig. 61], *EZ* will not intersect the other [opposite section] [IV.44]. Of course, *EZ* will not intersect *AB*; for, again, *AB* will not intersect the opposite sections *ABGD,EZ* at more points than four [IV.38]; neither will *GD* intersect *EZ*.

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[104] Since *ABE* meets *ABG*, it cannot, by IV.37, meet the opposite section of *ABG*, *i.e.*, *EZ*, at more points than two.

[105] With Heiberg, one must see the bracket sentence as useless and suspect.

If, as in the sixth drawing [see fig. 62], *ABGD* meets the other section at three points, *EZ* will intersect the other at one point only.

And we will say the same things as before for the remaining cases.

So, since what was proposed is clear in all possible configurations (διαστολάς), opposite sections will not meet opposite sections at more points than four.

Proposition 56

If opposite sections touch opposite sections at one point, they will not intersect at more than two other points.

Let there be opposite sections *AB,BG* and others, *D,EZ*,[106] let *BGD* touch *AB* at *B*, let them have their convexity turned oppositely, and, first, let *BGD* intersect *GD* at two points *G,D*, as in the first figure (σχήματος) [see fig. 63].

Indeed, since *BGD* cuts [*GD*] at two points having its convexity turned oppositely, *EZ* will not intersect *AB* [IV.41]. Again, since *BGD* touches *AB* at *B*, having its convexity turned oppositely, *EZ* will not intersect *GD* [IV.54]. Therefore, *EZ* will not intersect either of the sections *AB,GD*; therefore, [the sections] will meet at the two [points] *G,D*, only.

But let *BG* cut *GD* at one point *G*, as in the second figure [see fig. 64]. Therefore, *EZ* will not intersect *GD* [IV.54], whereas it will intersect *AB* at one point only.[107] For if *EZ* meets *AB* at two [points], *BG* will not intersect *GD* [IV.41]. But it was assumed that they meet at one [point].

If *BG* does not intersect section *D*, as in the third figure [see fig. 65], then, by what has been said [above], *EZ* will not intersect *D* [IV.54], whereas *EZ* will not intersect *AB* at more points than two [IV.37].

If the sections have their concavities in the same [direction] [see figs. 66, 67, 68], the same demonstrations will apply (ἁρμόσουσι).[108]

---

[106] In this formulation, we are to understand that *D* is the opposite section of *AB*, and *EZ* is the opposite section of *BG*. Since exactly the same wording is used in IV.57 but with the usual order of the sections (*i.e.*, the section and its opposite being paired together), I suspect the inversion here is a scribal error.

[107] The argument which follows shows only that *EZ* will intersect *AB* at one point *at most*. This, however, is sufficient for what Apollonius wants to show, namely, that when opposite sections touch at one point they can intersect again at two other points *at most*.

[108] The cases presented in figures 66, 67, 68 are proven by IV.48,49,50, respectively.

So, from the demonstrations, what was proposed is clear in all possible configurations.

Proposition 57

If opposite sections touch opposite sections at two [points], they will not intersect at another point.

Let there be opposite sections *AB,GD*, and others, *AG,EZ*, and, first, let them touch at *A,G*, as in the first figure [see fig. 69].

Indeed, since *AG* touches each of [the sections] *AB,GD* at points *A,G,EZ*, therefore, will intersect neither of [the sections] *AB,GD* [IV.51].

So, let them touch as in the second [figure] [see fig. 70]. It will be proved similarly that *GD* will not intersect *EZ* [IV.53].

So, let *GA* touch *AB* at *A* and [let] *D* [touch] *EZ* at *Z*, as in the third figure [see fig. 71].[109] Indeed, since *AG* touches *AB* having its convexities turned oppositely, *EZ* will not intersect *AB*. Again, since *ZD* touches *EZ,GA* will not intersect *DZ*.

If *AG* touches *AB* at *A*, while *EG* touches *GD* at *G*, having their concavities in the same [direction], as in the fourth figure [see fig. 72], they will not intersect at another point [IV.52]. *EZ* will not even intersect *AB*.

So, from the demonstrations, what was proposed is clear in all possible configurations.

[109] In fact, this case cannot occur by IV.54.

Diagrams

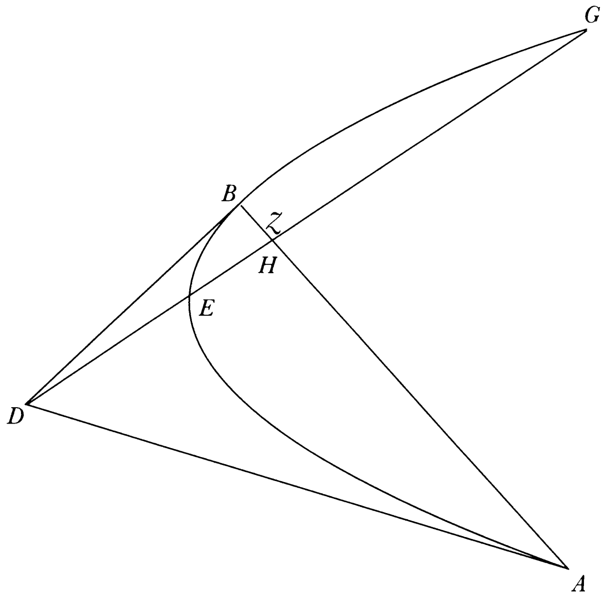


Fig. 1

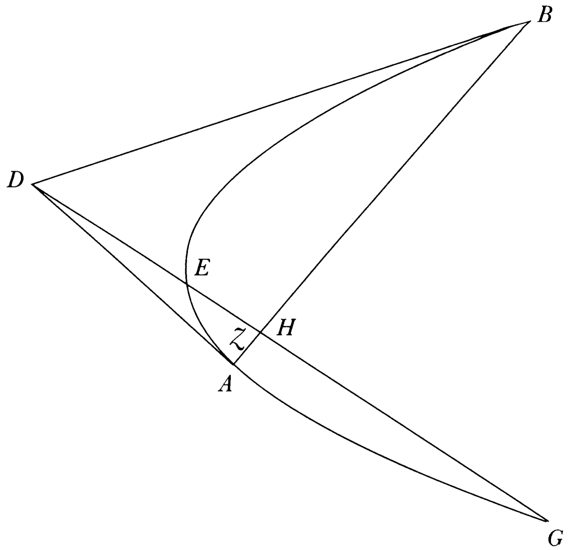


Fig. 2

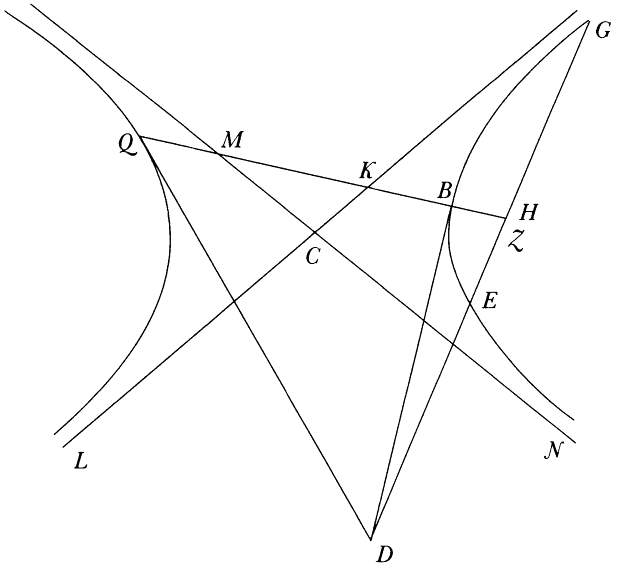


Fig. 3

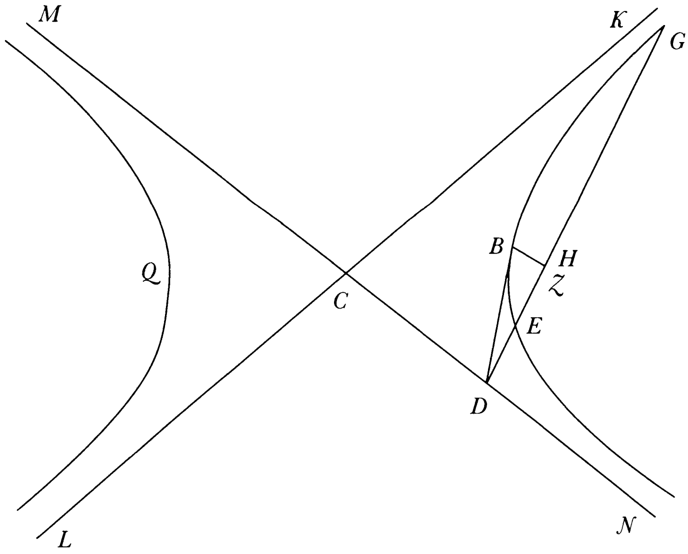


Fig. 4

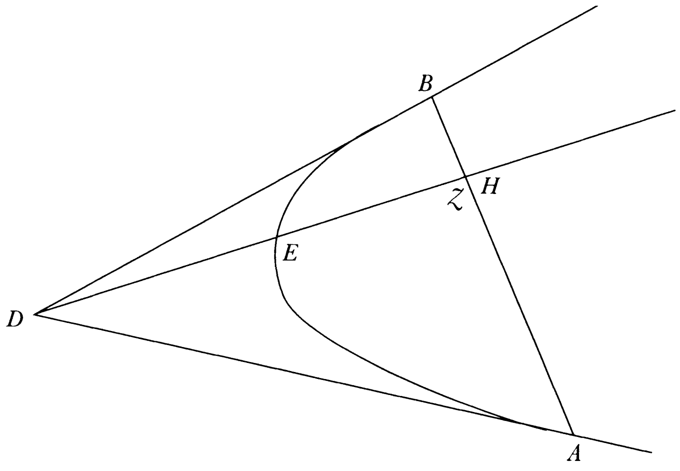


Fig. 5

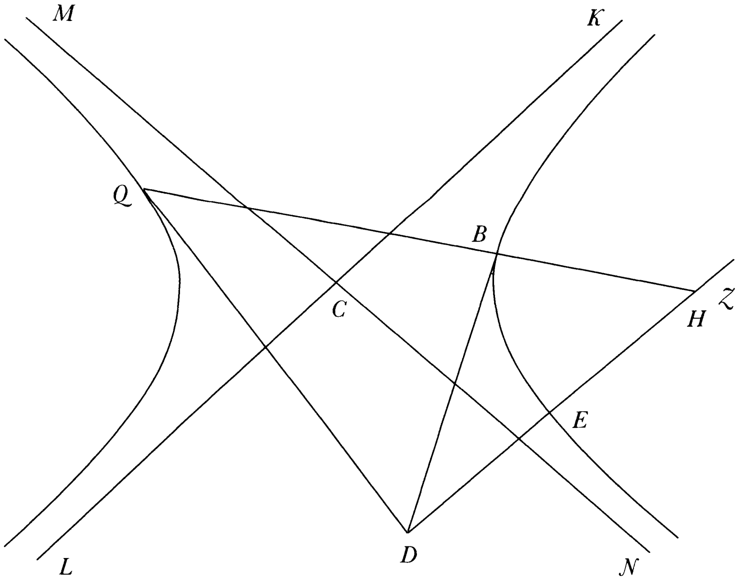


Fig. 6

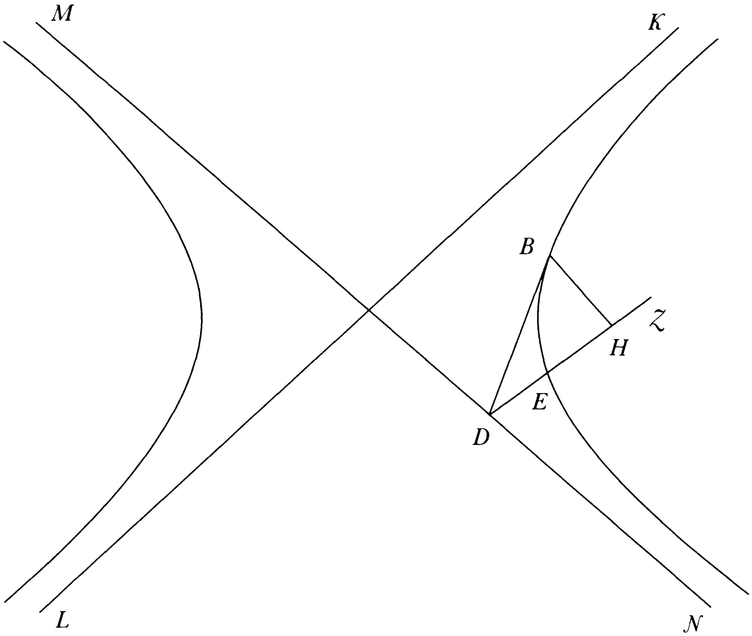


Fig. 7

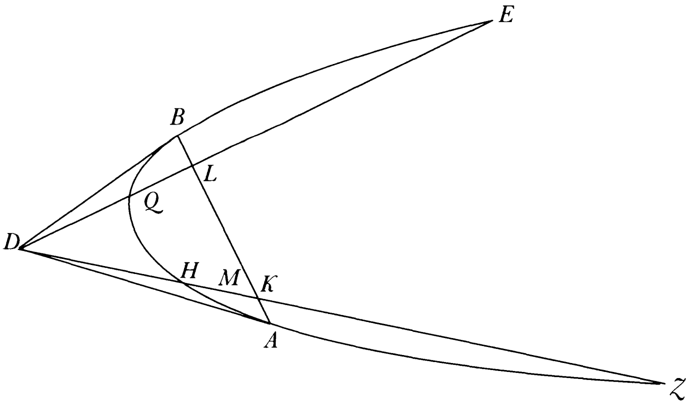


Fig. 8

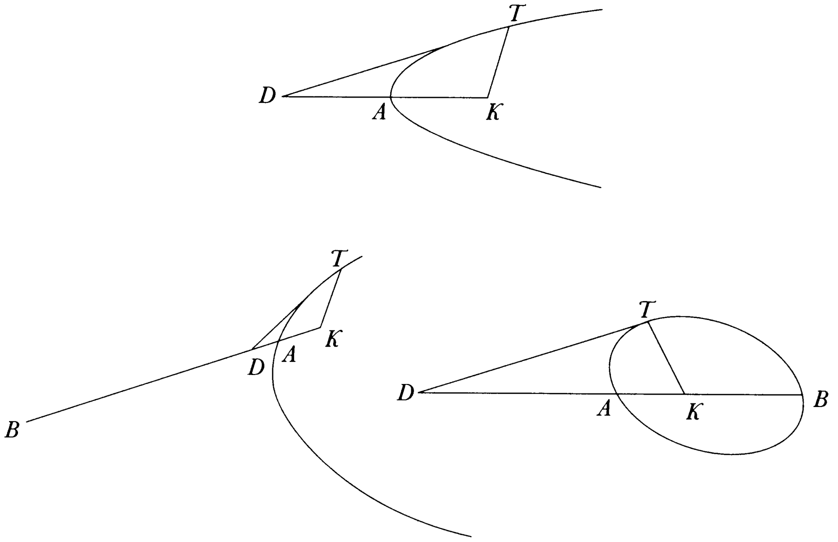


Fig. 9

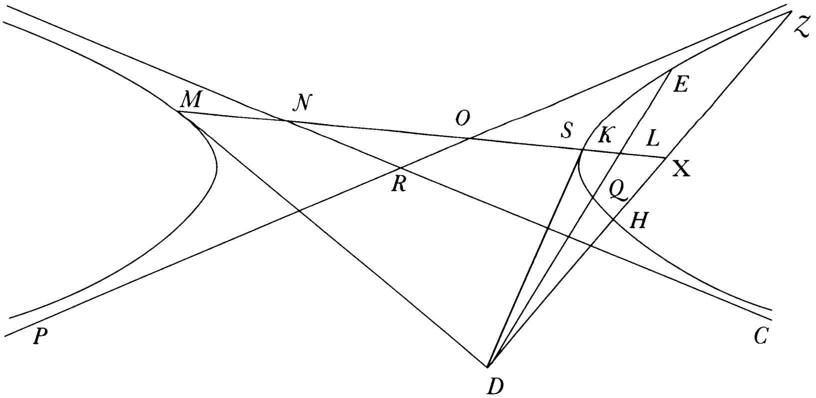


Fig. 10

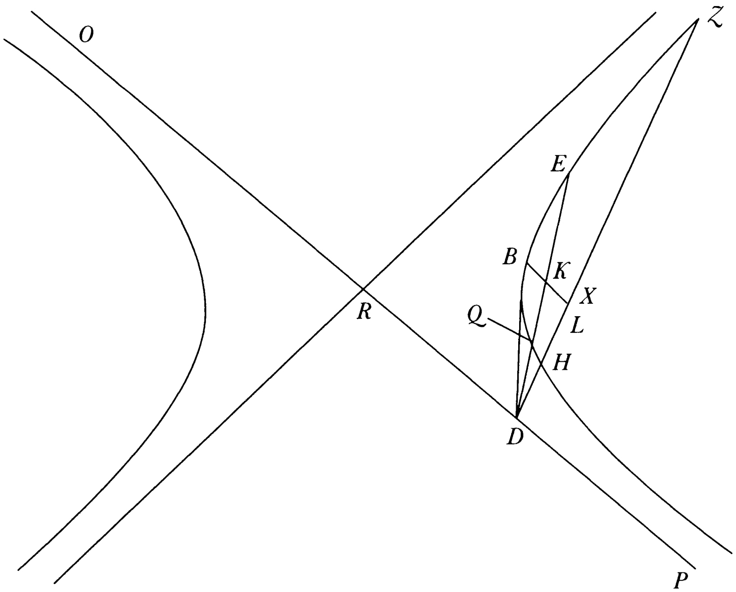
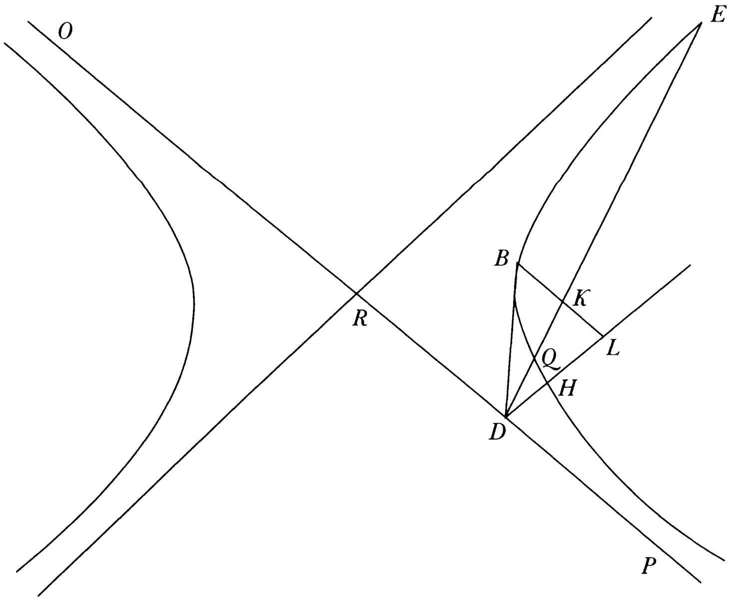


Fig. 11



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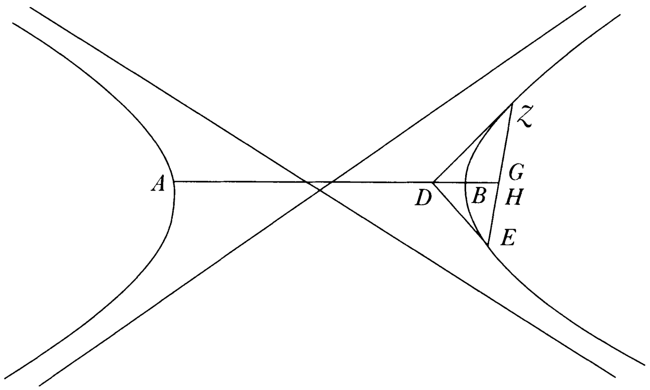
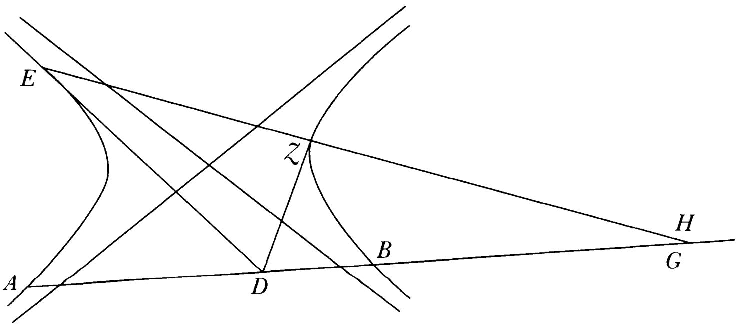


Fig. 13



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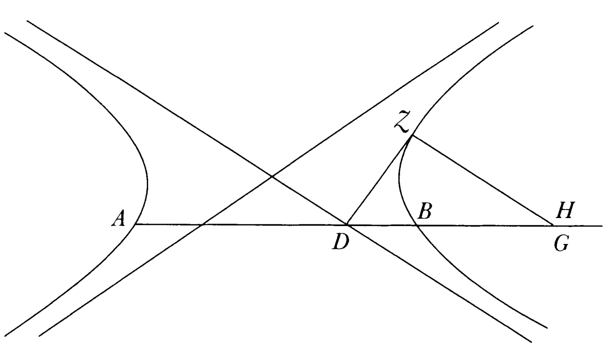

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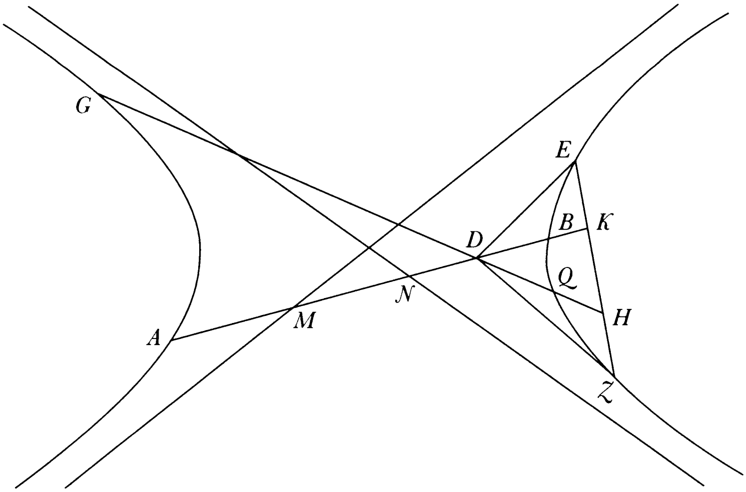


Fig. 16

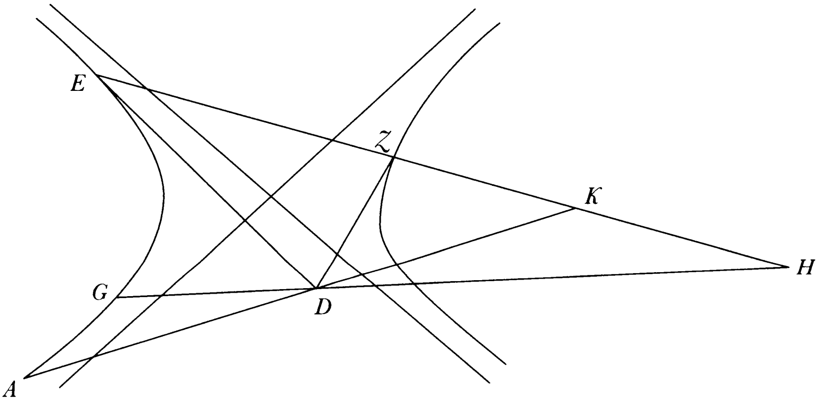
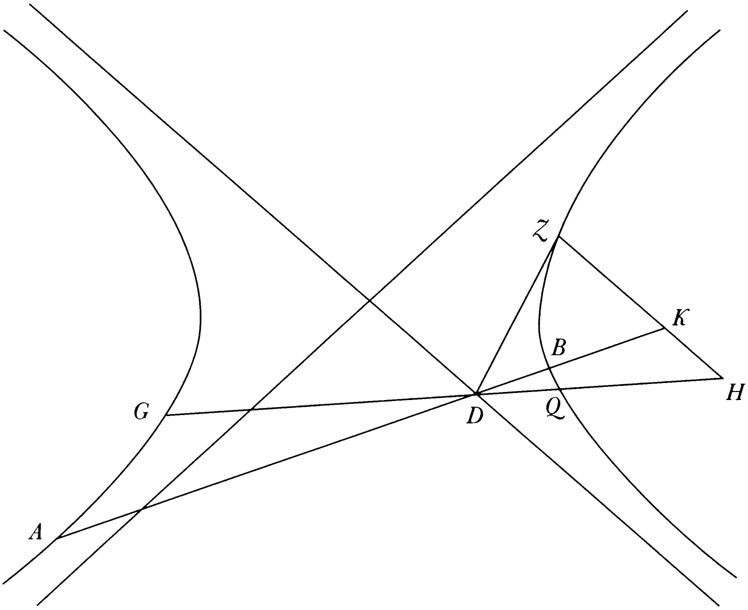


Fig. 17



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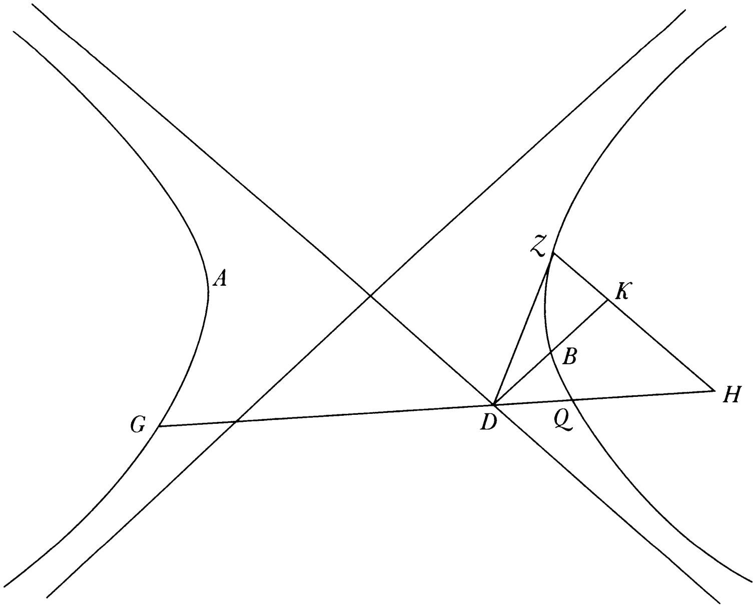

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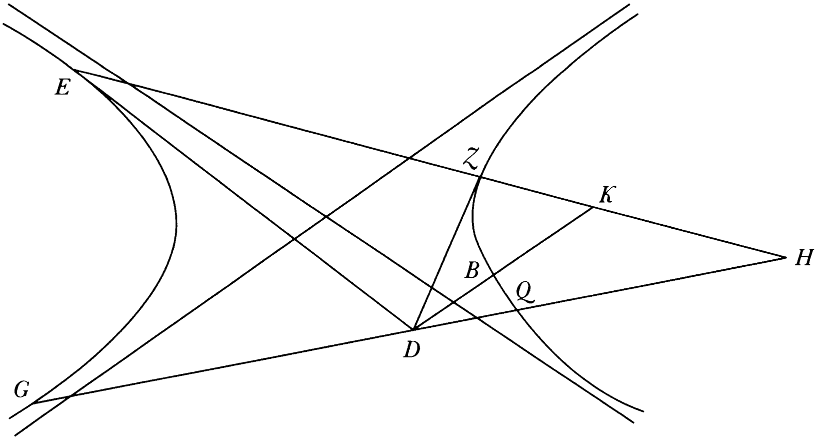

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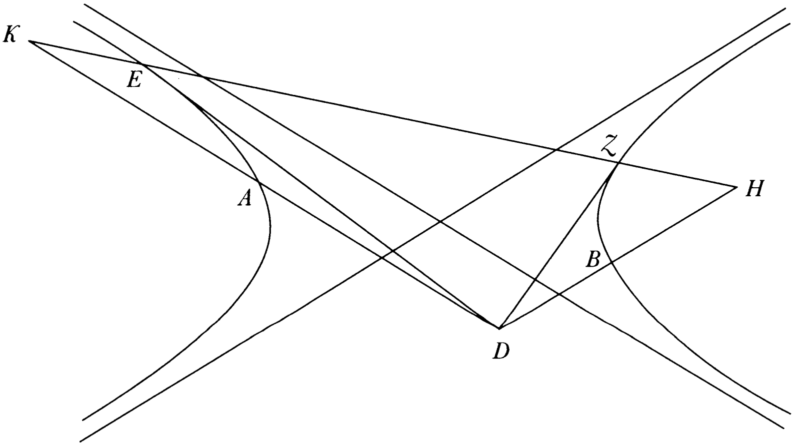

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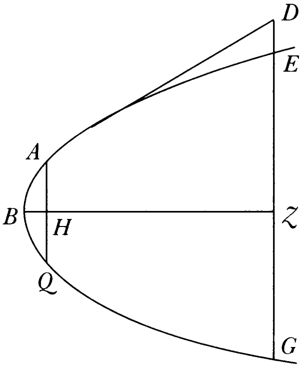

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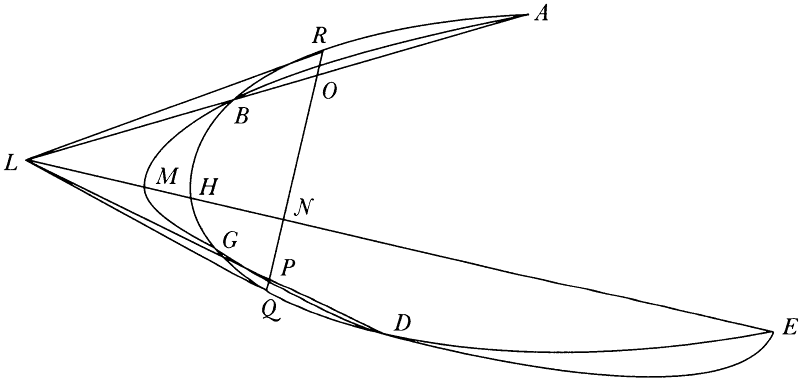

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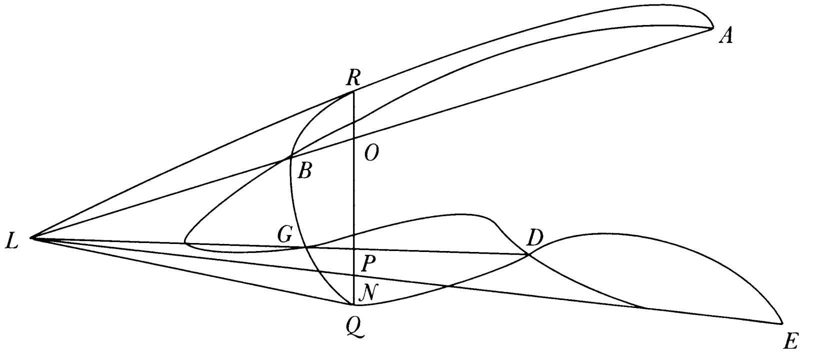

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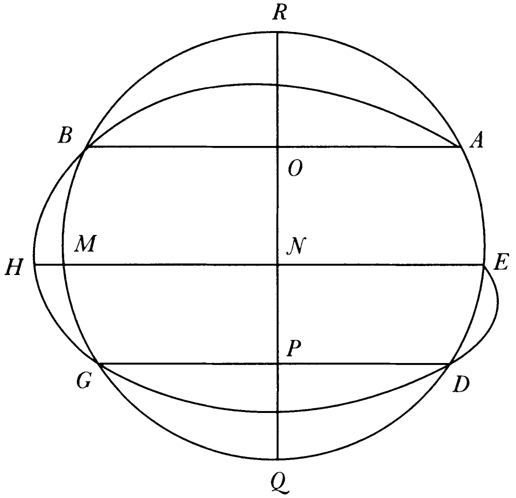

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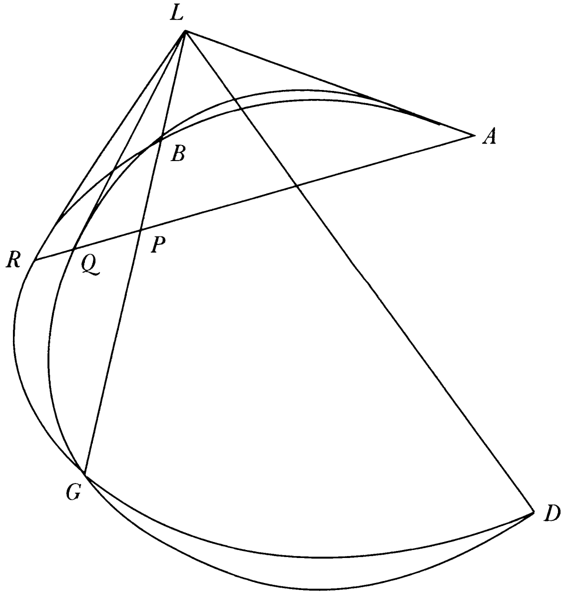

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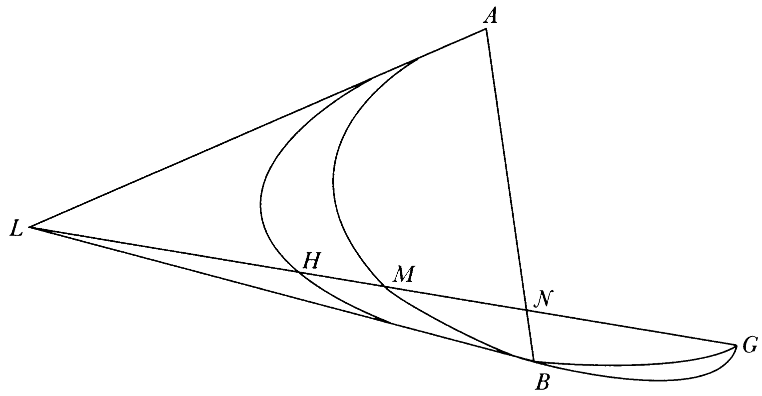

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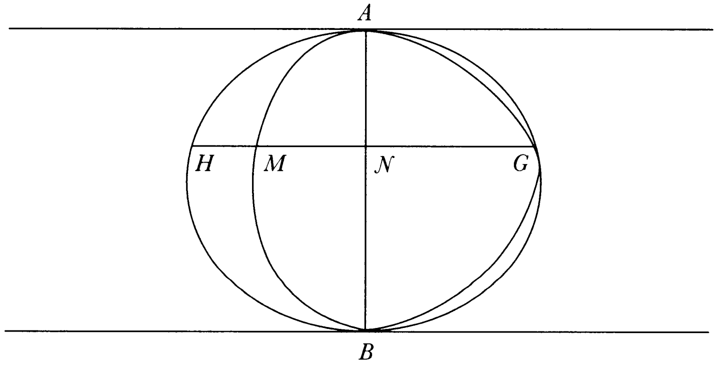

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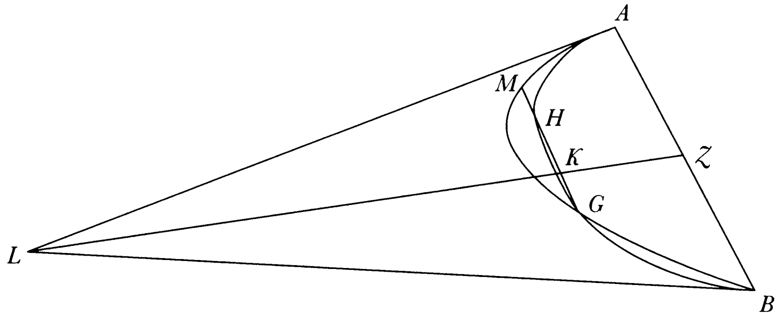

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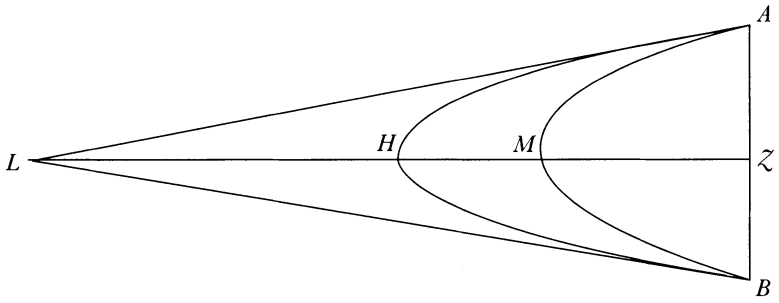

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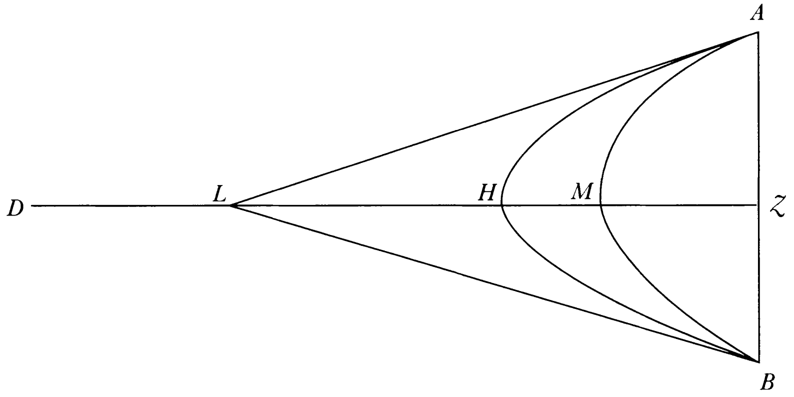

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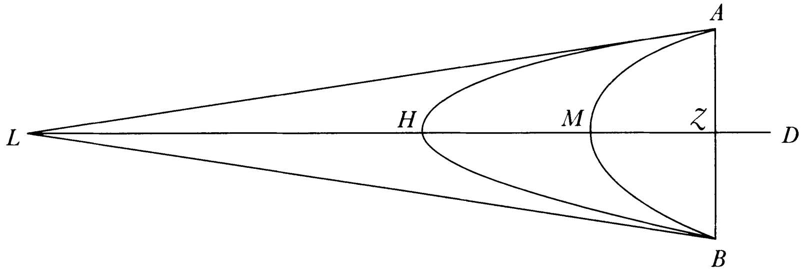

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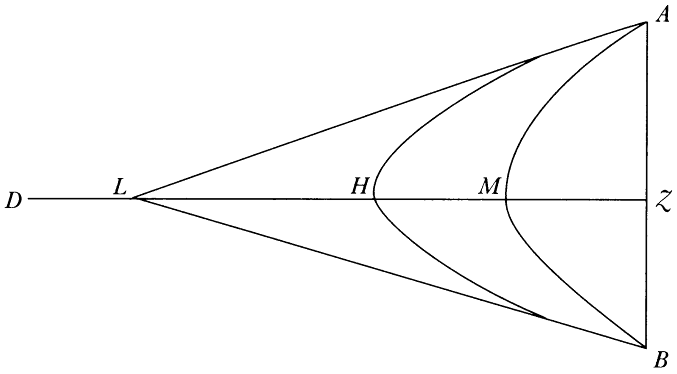

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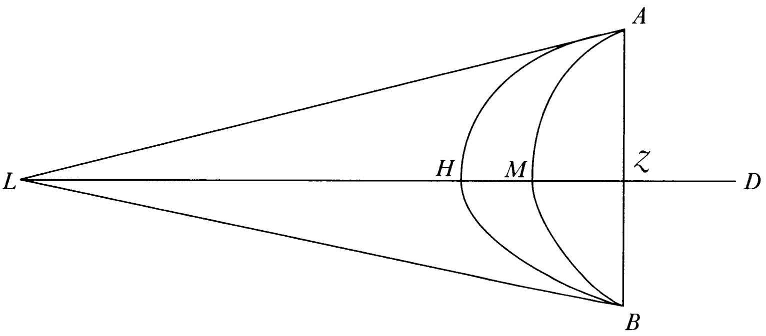

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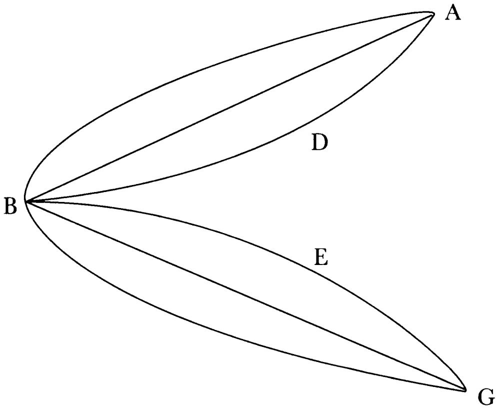

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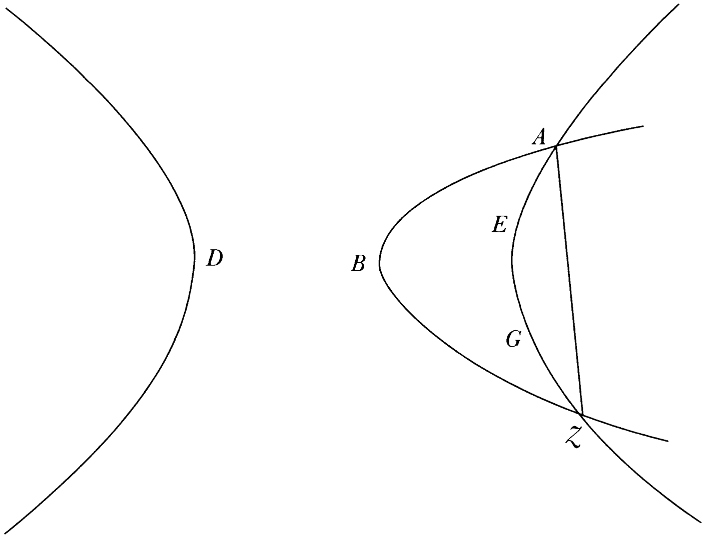

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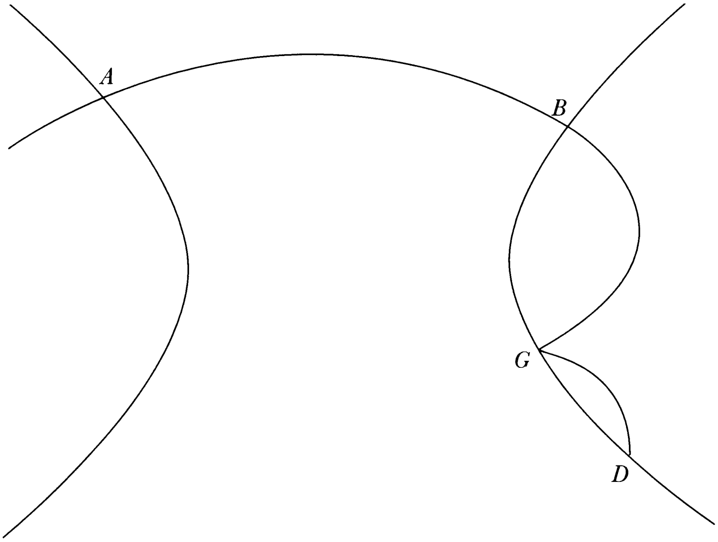

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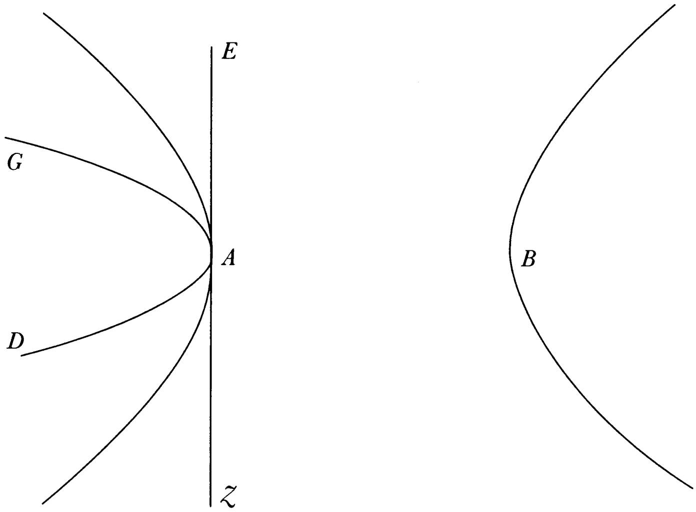

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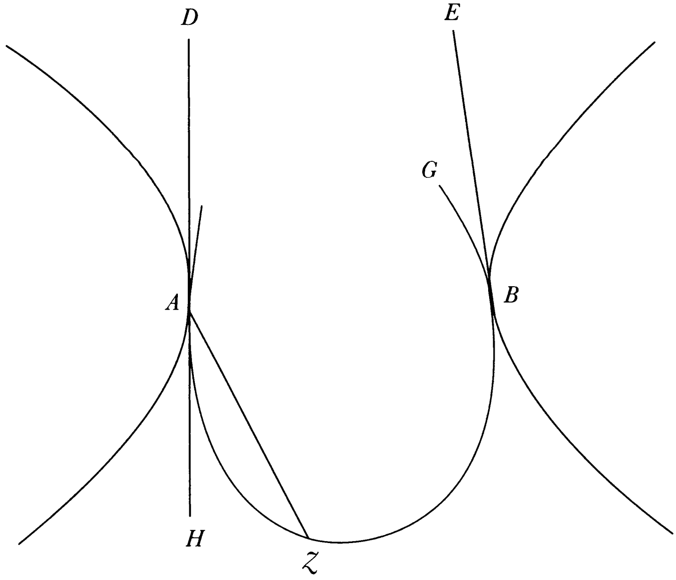

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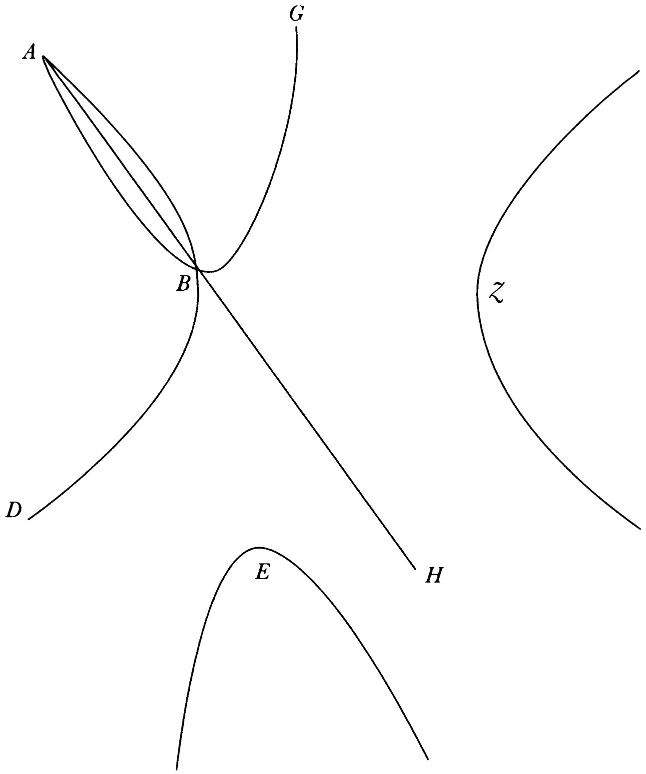

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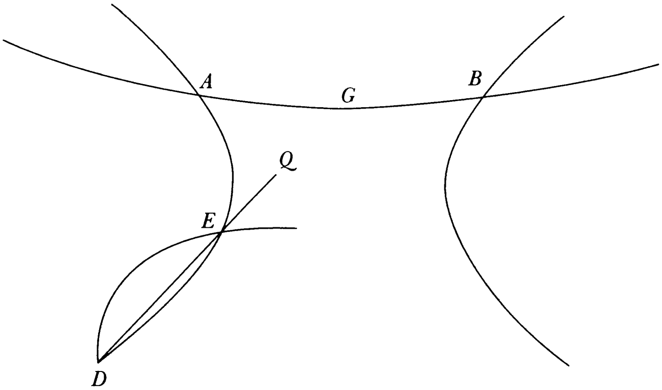

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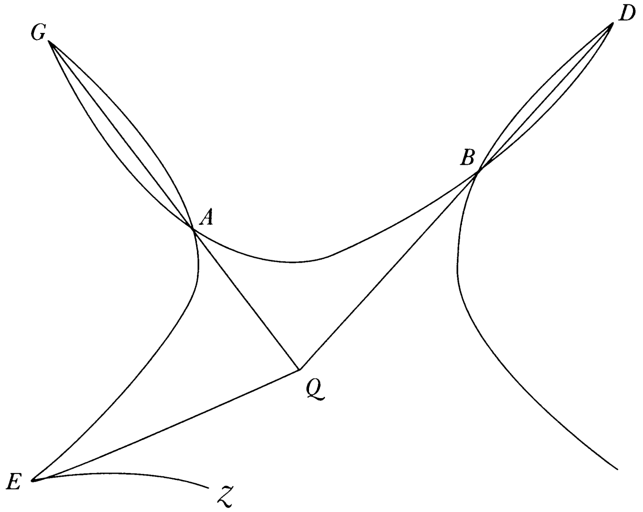

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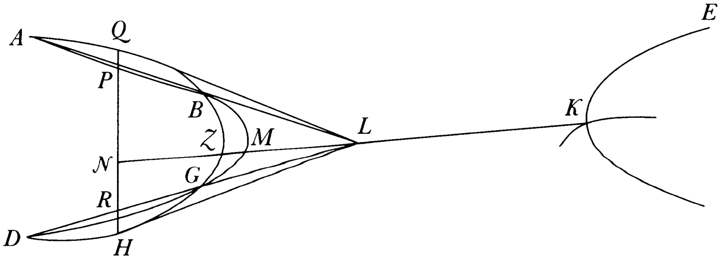

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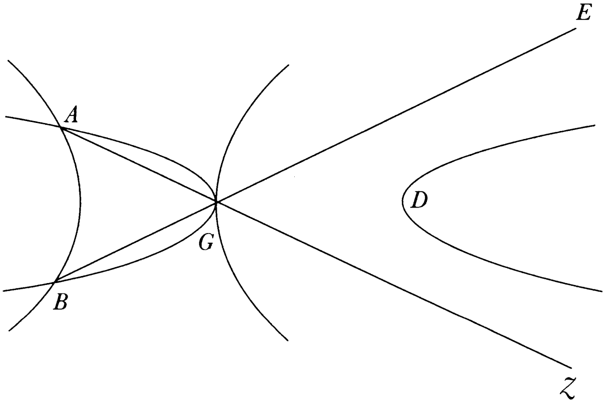

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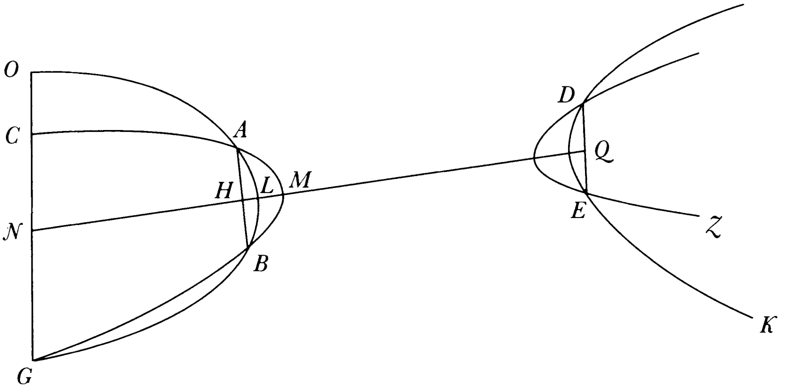

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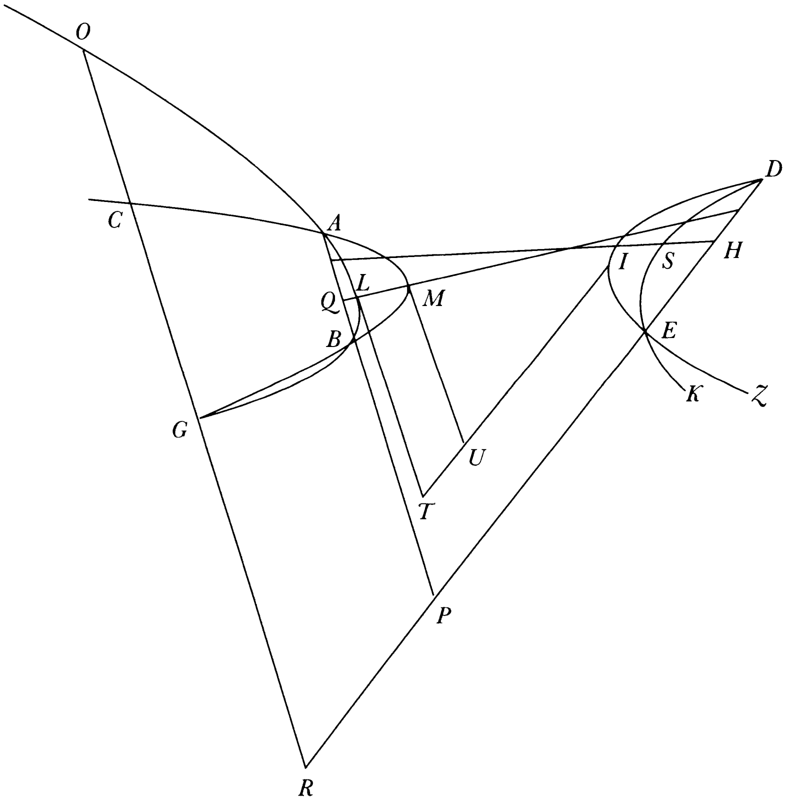

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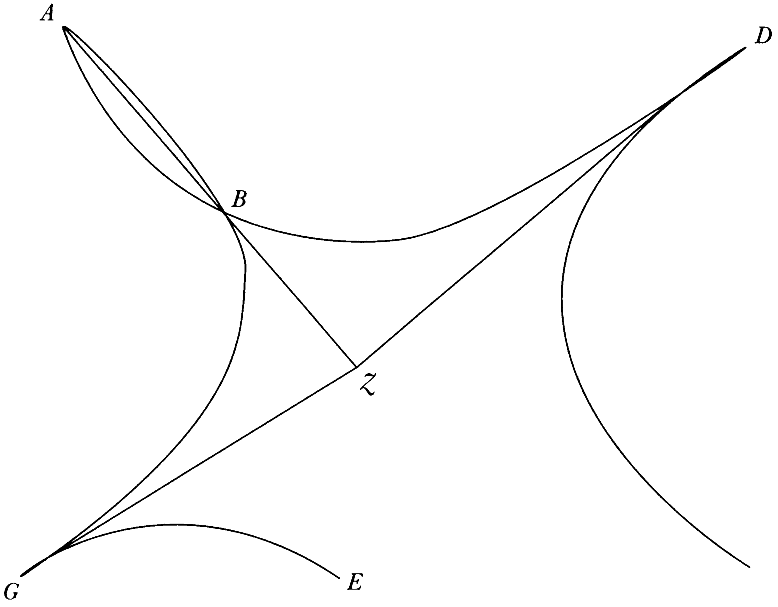

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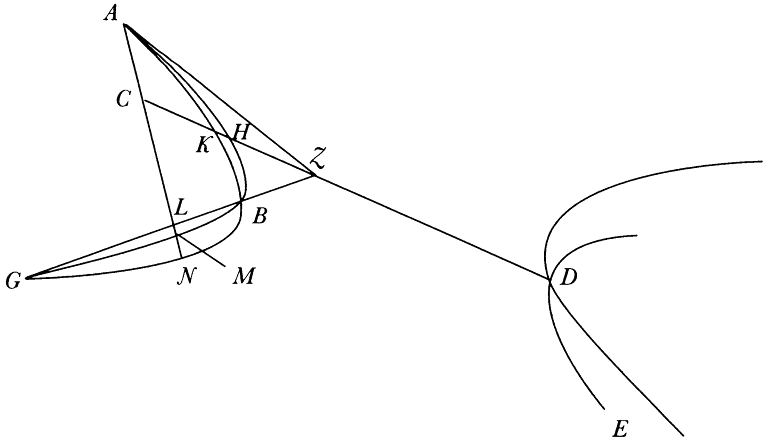

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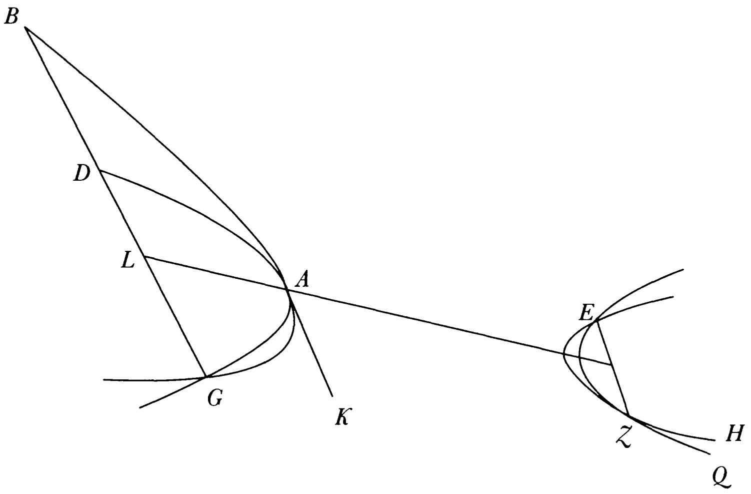

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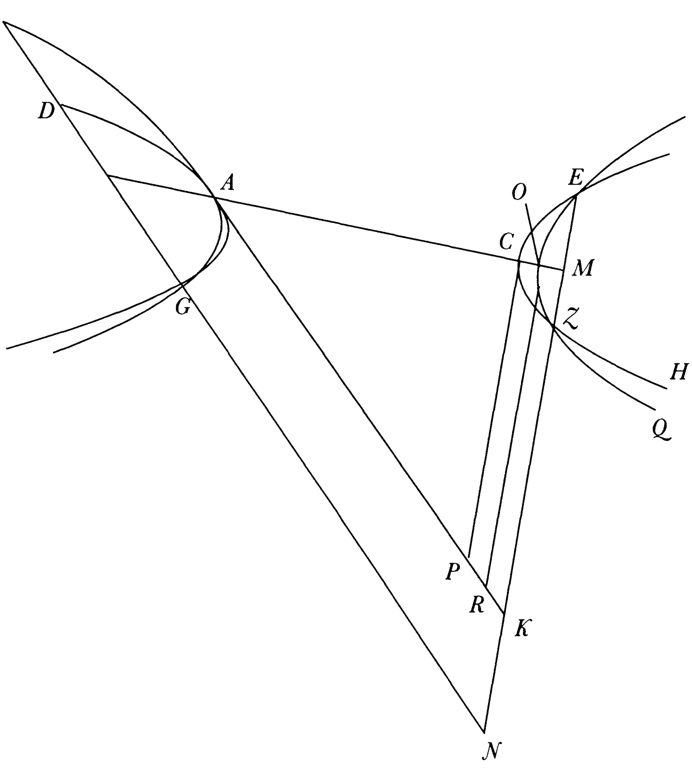


Fig. 50

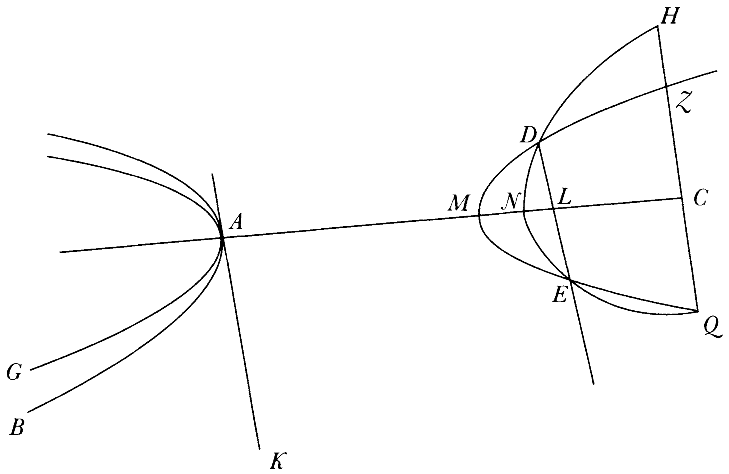
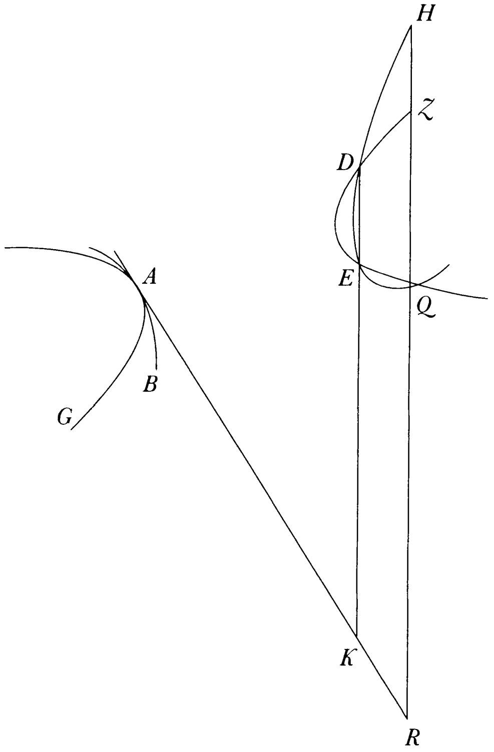


Fig. 51



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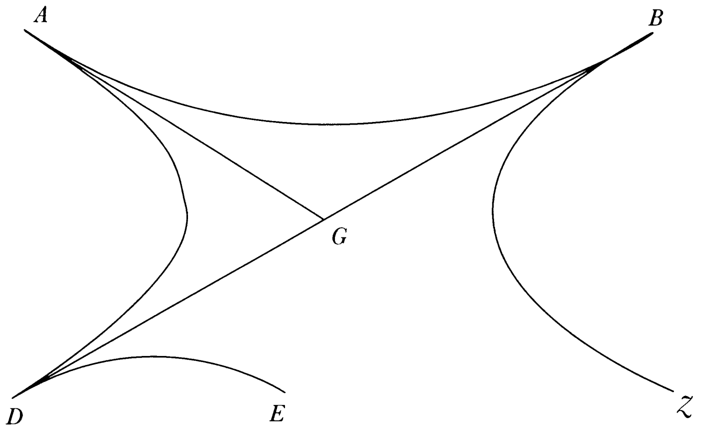

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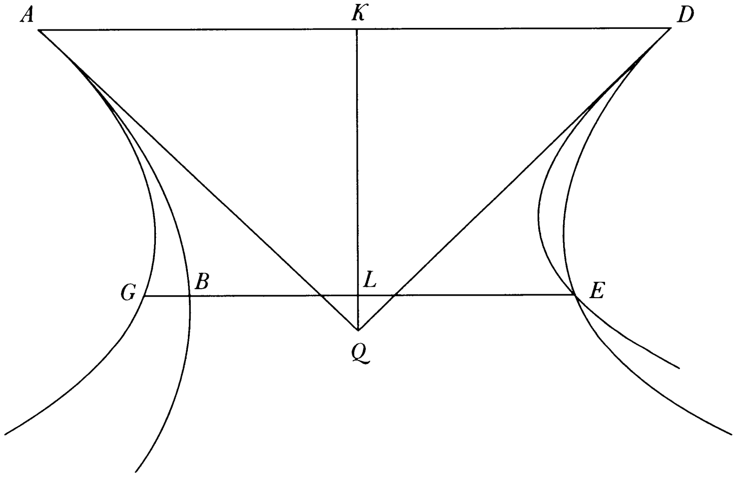
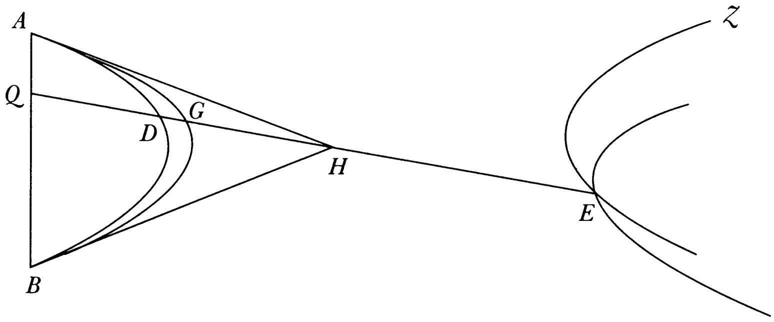


Fig. 54



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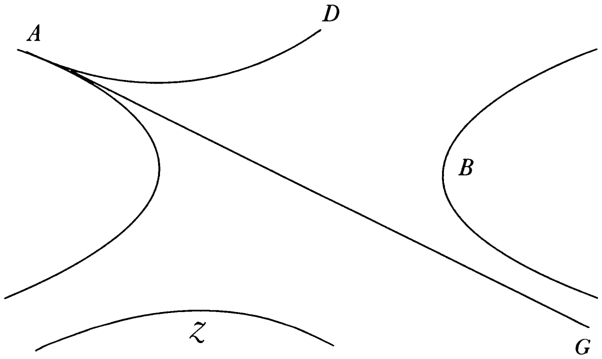

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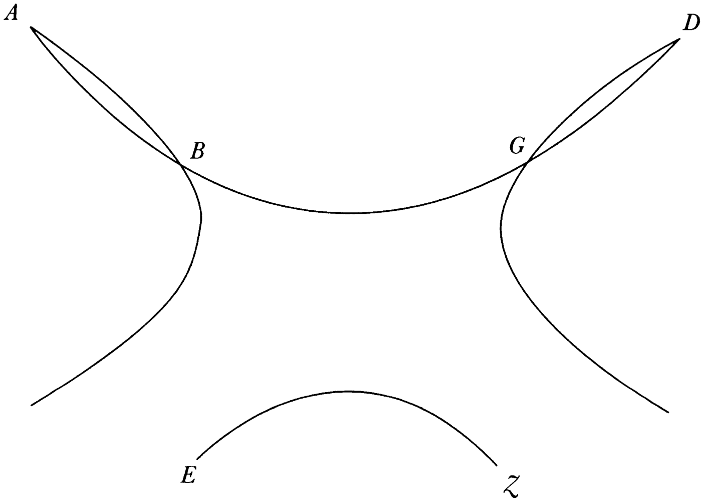

Fig. 57

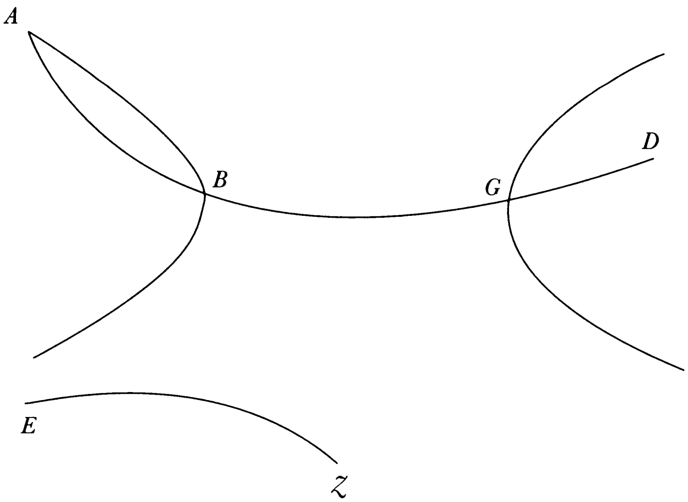

Fig. 58

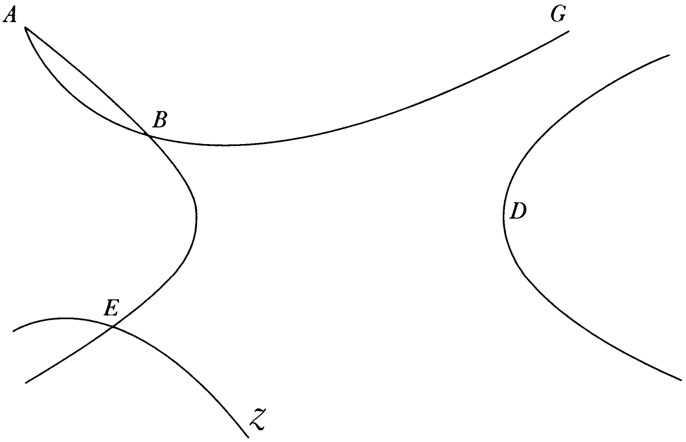

Fig. 59

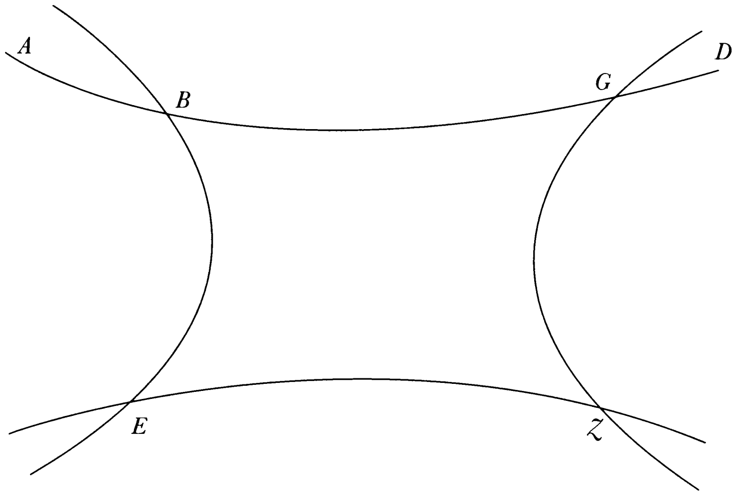

Fig. 60

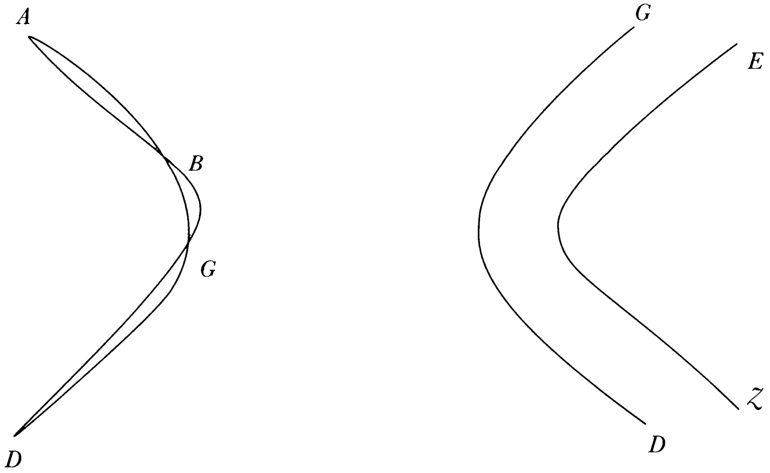

Fig. 61

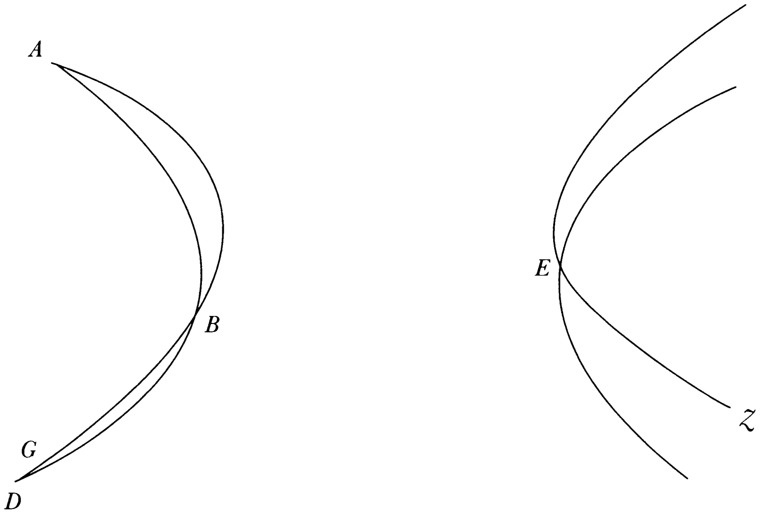

Fig. 62

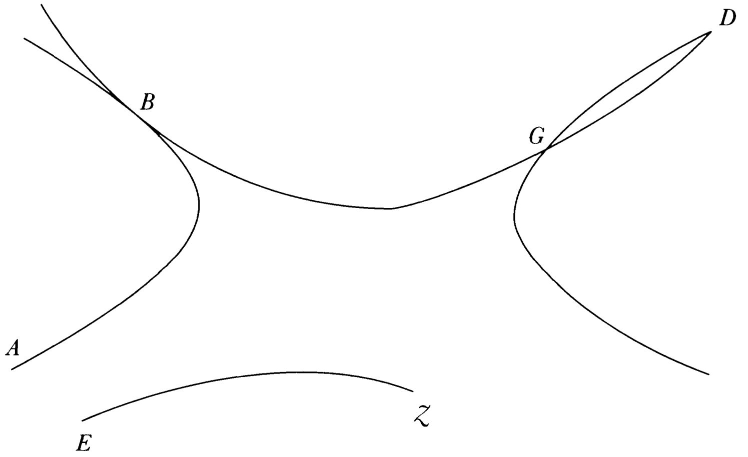

Fig. 63

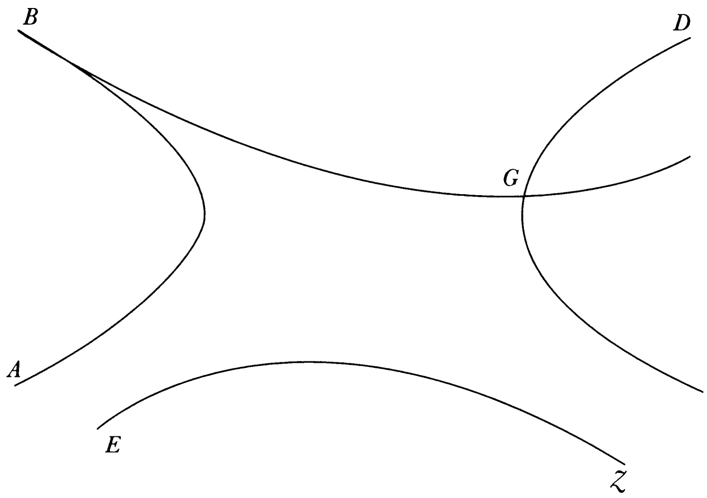

Fig. 64

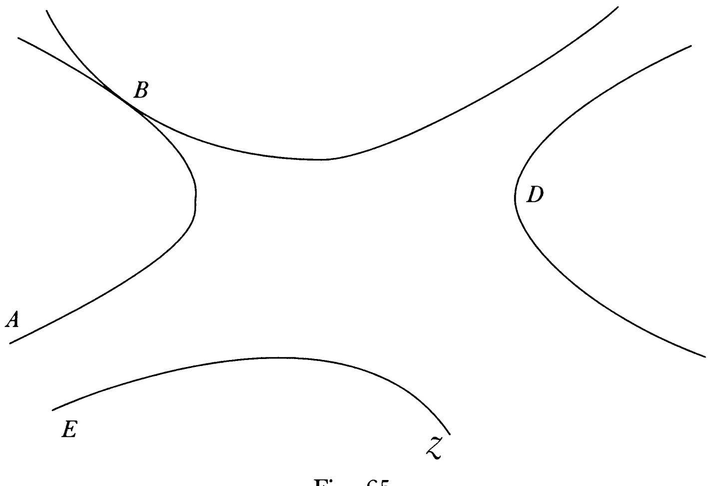

Fig. 65

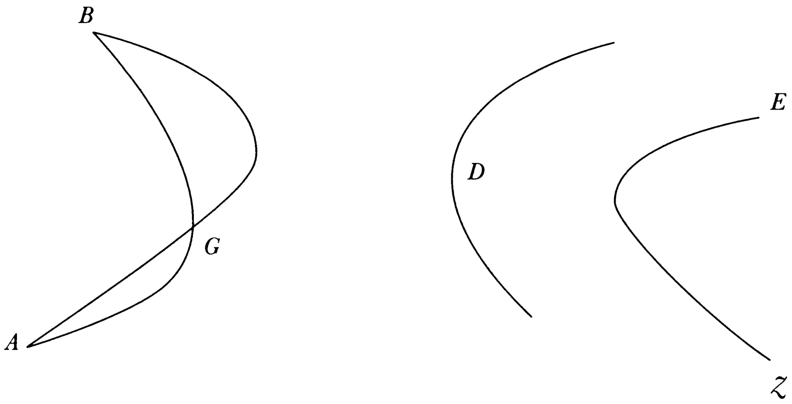

Fig. 66

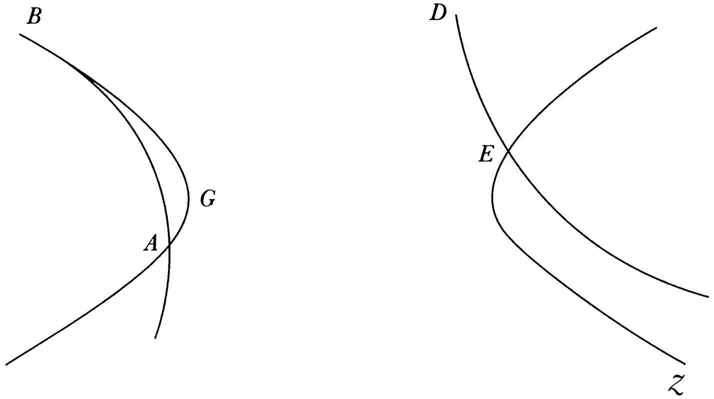

Fig. 67

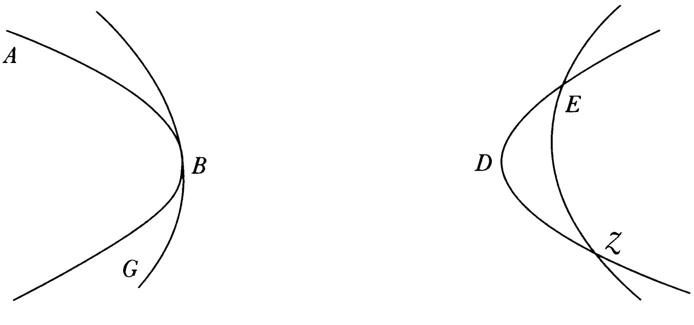

Fig. 68

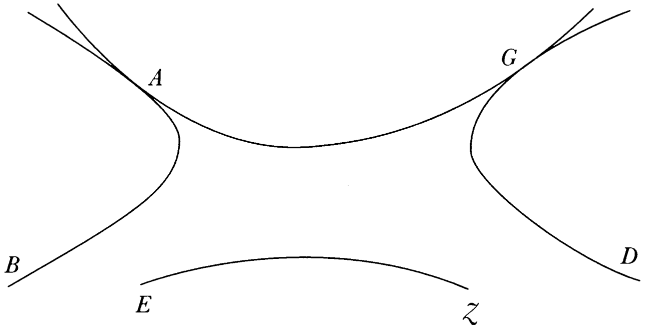

Fig. 69

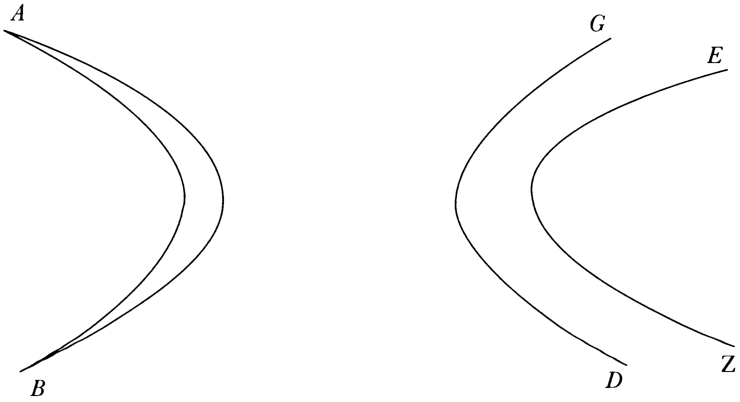

Fig. 70

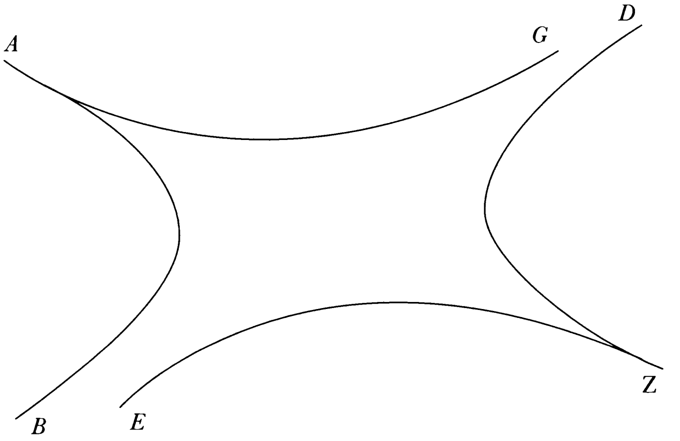

Fig. 71

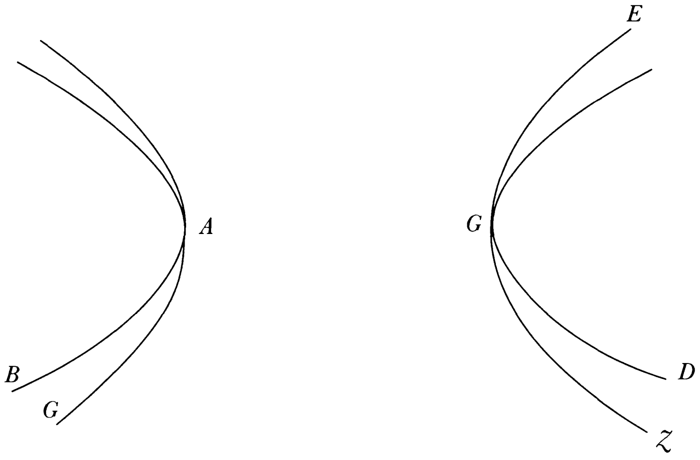

Fig. 72

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